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FOUNDATIONS, THEORY OF SETS, LOGIC

★**Church, Alonzo. Introduction to mathematical logic. Vol. I.** Princeton University Press, Princeton, N. J., 1956. x+376 pp. $7.50.

This is the long awaited, revised and very much enlarged edition of "Introduction to mathematical logic", Part I [Princeton, 1944; MR 6, 29]. It is intended to be used as a textbook by students with substantial mathematical background and also, within limitations, as a reference work. In both roles it seems to succeed admirably. To help its use as a reference, there is an exhaustive index of definitions which also acts as a subject-index, an index of authors, and 590 footnotes numbered consecutively throughout this volume. The footnotes are an integral part of the book, adding greatly to the subject matter through additional references, insight, background material, examples and comparisons. As a reference and guide to relerant literature this book is much easier to use than Hilbert and Bernays' Grundlagen der Mathematik [2 vols., Springer, Berlin, 1934, 1939], partly because of the excellent index system and partly because theorems are stated explicitly rather than as part of the text. In addition of course, Church takes into account the developments of the last few years. However it should be emphasized that the book by Hilbert-Bernays covers far more material and is not likely to lose its place in the literature which it has held so securely for about twenty years. Also some important results which are proved in detail in Hilbert-Bernays are here given as exercises (cf. Herbrand's theorem, Hilbert-Bernays vol. 2 p. 149, Church p. 270) and many a student is likely to turn to Hilbert-Bernays for further help. Exercises are numerous. A few of these are elementary, but others sketch many important extensions of the material presentep in the text.

The Introduction is a chapter by itself leading up to a brief discussion of the logistic method, syntax and semantics. First Church shows the need for some formalization of the language used in logic. Then follows a discussion of names, variables, constants and functions. The treatment given here of sentences as names of a special sort is due to Frege. Following the introduction of the notions of propositions, propositional functions, operators and quantifiers, the section on the logistic method introduces the concepts of object- and meta-languages, logistic systems, formulas, axioms, postulates, rules of inference, theorems, and the axiomatic method. Postulates are axioms of mathematical content and are thought of as determining a special branch of mathematics. Proofs are required to be effective. Church distinguishes between the elementary syntax of a language and theoretical syntax. The first is used to set up a logistic system and is a restricted formalized meta-language. Theoretical syntax deals with the general mathematical theory of logistic systems apart from their interpretation, and criteria of effectiveness may or may not be abandoned. In this volume, the object language is formalized while the theoretical syntax is treated informally. Semantics is the study of the interpretation of a language as an interpretation. Tarski's work on the concept of truth and the possibility of finding syntactical concepts which are equivalent in extension to various semantical ones such as the property of being a true sentence or of denoting, are also discussed.

Chapters I and II concern the propositional calculus. First Church presents the calculus P_1 which is a formalization using $[, \supset,]$, $/$ and an infinite list of variables as primitive symbols and has 3 axioms with modus ponens and substitution as rules of inference. Chapter I gives a rather complete discussion of P_1, tautologies and the decision problem, duality, consistency and completeness and independence of the axioms and rules of inference of P_1. There are three syntactic definitions of consistency (p. 108) and three corresponding definitions of completeness. In Chapter II there are alternative formulations of the classical propositional calculus as well as formulations of a number of other calculi. Some of these are found in the exercises, many of which are not at all easy. Chapter II concludes with historical notes which serve not only to put the subject matter into perspective, but also give many leads for further study.

Chapter III deals with the first order functional calculi as based on 5 axiom schemata with modus ponens and substitution as rules of inference. These calculi are consistent, but not complete in any of the various senses of completeness of Chapter I. Church proves various further theorems, theorem schemata and derived rules of these calculi, as for instance various rules of substitution and the deduction theorem. Intuitionistic systems and Gentzen's formulation of the first order calculi are described in the exercises.

The pure first order functional calculus is given an alternative formulation in Chapter IV and the independence of the various rules and axioms is discussed. This is followed by sections on the Skolem normal form and its relation to validity and satisfiability, leading to completeness theorems and the Skolem-Löwenheim theorem. Both Gödel's and Henkin's proofs for completeness are given. The two sections on decision problems and reductions of the decision problem include many of the most important results either in the text or exercises. (For example see exercise 46.23, p. 270 for Herbrand's theorem.)

Chapter V deals primarily with second order functional calculi, although as pointed out, many of the methods and results also hold for higher calculi or can be generalized fairly easily, so that much of the material also forms a basis for the theory of higher functional calculi and as background for the theory of types. After formulating the second order functional calculus, and showing how to define equality for individuals, Church returns to the question of consistency and completeness, proving Henkin's extension of the completeness theorem. In an

interesting digression on postulate theory it is pointed out that one may look at postulates of a mathematical theory from two different points of view: they may be axioms added to the logical axioms of a logistic system or they may serve to describe the models of the theory. Some of the axioms which one may wish to adjoin are expressible in the pure second order functional calculus without added functional or individual constants, as for instance some axioms which state that the individuals of the domain way be well ordered and various axioms of infinity. The chapter ends with the problem of predicative vs. impredicative definitions which leads to a discussion of ramified calculi and axioms of reducibility.

Even a brief glance at the tentative table of contents of volume II shows that it will likely be even richer in content than volume I. *I. N. Gál* (Ithaca, N.Y.).

Issman, Samuel. La notion de preuve. Methodos **7** (1955), 201–208.

This paper considers the empirical problem of convincing a person, rather than the logical problem of proving to a person. Various classes of propositions are discarded as "useless". *C. C. Torrance.*

Nicolescu, M. The notion of equivalence and its importance in mathematics. Gaz. Mat. Fiz. Ser. A. **8** (1956), 337–345. (Romanian)

A lecture delivered for pupils of secondary schools. *B. Germansky* (Jerusalem).

van Rootselaar, B. A remark on Brouwermeasurable functions. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 579–580.

Proof of the following theorems on functions defined on a subspecies of a closed interval are given. 1. Any Brouwer-measurable function is almost bounded. 2. The sum of two Brouwer-measurable functions with the same domain is Brouwer-measurable. [See A. Heyting, Intuitionism, North-Holland Publ. Co., Amsterdam, 1956, p. 85; MR **17**, 698.] *A. Heyting* (Amsterdam).

★ **Tarski, Alfred. Ordinal algebras, with appendices by Chen-Chung Chang and Bjarni Jónsson.** North-Holland Publishing Co., Amsterdam, 1956. i+133 pp. $3.60.

An ordinal algebra is an additively written, but not ordinarily commutative, associative system, equipped with a suitably axiomatized operation of simply infinite addition, $\sum_1^\infty a_\nu$, and an operation of conversion, a^*. (Infinite addition is characterized and employed "only insofar as [it is] involved in the study of finite addition".) The additive theory of order types provides a familiar application, although far from exhausting the interest of the theory. The theory of ordinal algebras differs from that presented in the author's "Cardinal algebras" [Oxford, New York, 1949; MR **10**, 686] principally in the non-commutativity of addition.

Chapter 1 constitutes an axiomatic development of the arithmetic of ordinal algebra. In particular, it provides a formal algebraic study of various arguments from the theory of order types, and of the rôle of the well-ordering axiom in such arguments. An ordinal algebra is, in paraphrase, an associative system under $a+b$, with neutral element 0; conversion a^* is an involution satisfying $(a+b)^*=b^*+a^*$; $\sum$ is subject to three axioms, as follows. A weak associativity axiom requires that $a_1+\sum_1^\infty a_{\nu+1}=\sum_1^\infty a_\nu$. The directed refinement postulate requires that, if $b+c=\sum_1^\infty a$, then, for some n, $a_n=b'+c'$ and $b=\sum_1^{n-1} a+b'$, $c=c'+\sum_{n+1}^\infty a$. The remainder postulate [cf. Cardinal algebras, p. 5] requires that, if $r_n=b_n+r_{n+1}$ for all n, then there exist c such that $r_n=\sum_{n+1}^\infty b+c$ for all n. Representative theorems are the following. (1.49) If $am=bn$ (m, n positive integers, obvious meaning), then $a+b=b+a$. (1.50: Euclid) If $am=bn$ and m, n are relatively prime, then, for some c, $a=cn$, $b=cm$.

Chapter 2 deals with the addition of isomorphism types of (reflexive) binary relations. If R_ν are relations defined on disjoint sets F_ν, $\nu=1, 2, \cdots$, the relation $\sum_1^\infty R$ is defined to relate x to y just in case $xR_\nu y$ for some ν, or else x is in F_μ and y in F_ν where $\mu\leqq\nu$. The types of reflexive relations, under this addition, are shown to constitute an ordinal algebra. More generally, $\sum_S R_\sigma$ is defined similarly with the relation $\mu\leqq\nu$ on the index set of positive integers replaced by an arbitrary relation S on a set $\{\sigma\}$. (2.12: Jónsson-Tarski) Every reflexive relation R has a unique representation as $\sum_S R_\sigma$ where S is a simple-ordering relation and each R_σ is indecomposable (not $R=P+Q$ non-trivially).

Appendix A, by Chang, contains results bearing on the general question of when a linear relation in a and b implies $a+b=b+a$.

Appendix B, by Jónsson, deals with unique decomposition theorems for relation types. The main result establishes a unique decomposition $R=\sum_S R_\sigma$ for S restricted to a suitably characterized class K of relations, and each R_σ indecomposable relative to the class K. For K the class of order types, this gives 2.12, above. For K the class of identity relations (on various domains), this gives the theory of cardinal addition (R the union of the R_σ as sets of couples). For K the class of universal relations, it gives the theory of 'square addition'. It is shown that the last two of these theories, unlike the first, are trivial in the sense that they are commutative and reduce to the addition of cardinal numbers.

R. C. Lyndon (Berkeley, Calif.).

Orey, Steven. On ω-consistency and related properties. J. Symb. Logic **21** (1956), 246–252.

The author gives necessary and sufficient conditions, in terms of models, for a large class of logics to be ω-consistent. He also shows that there is an ω-consistent logic which has no model of a certain kind. *A. Rose.*

Bing, Kurt. On simplifying truth-functional formulas. J. Symb. Logic **21** (1956), 253–254.

From a result of Samson and Mills [Communications Laboratory, Electronics Research Directorate, Air Force Cambridge Research Center, Cambridge, Mass., Tech. Rep. **54–21** (1954); MR **17**, 436], Bing deduces some results which are of use for simplifying truth-functional formulae. *A. Rose* (Nottingham).

Hermes, Hans. Über die gegenwärtige Lage der mathematischen Logik und Grundlagenforschung. Jber. Deutsch. Math. Verein. **59** (1956), Abt. 1, 49–69.

Report on recent work. Stress is laid on investigations in semantics. *A. Heyting* (Amsterdam).

Schröter, Karl. Die Unabhängigkeit der elementaren prädikatenlogischen Schlussregeln. Z. Math. Logik Grundlagen Math. **2** (1956), 218–227.

H. Scholz has introduced an axiomatization of the elementary logic of predicates in which all laws of this system are obtained, starting from the set X of all formu-

las valid in sentential logic, by applying seven rules of inference, S_1-S_7. The author proves the independence of each rule S_k by the following method: (1) All laws obtainable from X by applying only rules S_1-S_{k-1}, $S_{k+1}-S_7$ are shown to have a certain structural property P_k; (2) a certain law H_k is exhibited which does not have the property P_k. *E. W. Beth* (Amsterdam).

Rose, Alan. Formalisation du calcul propositionnel implicatif à m valeurs de Łukasiewicz. C. R. Acad. Sci. Paris **243** (1956), 1263–1264.

The author sketches a proof that an axiom set for the m-valued Łukasiewicz implicational propositional calculus can be obtained by adding a single axiom A5, described below, to the set of four axioms A1-A4 shown by the author to be an axiom set for the $\aleph_0$-valued implicational calculus [C. R. Acad. Sci. Paris 243, 1183–1185 (1956); MR **18**, 271]. If the functors $\rightarrow_i$ $(i=1, 2, \cdots)$ are defined by:

$$P\rightarrow_1 Q=P\rightarrow Q \text{ and } P\rightarrow_{i+1} Q=P\rightarrow(P\rightarrow_i Q),$$

then A5 is $(P\rightarrow_{m-1} Q)\vee P$.

E. Mendelson (Cambridge, Mass.).

See also: Novotný, p. 636; Mycielski, p. 661; Borovskiĭ, p. 664; Dolbeault, p. 670; Astrahan, Housman, Jacobs, Mayer and Thomas, p. 678; Novák, p. 679; Vâlcovici, p. 684.

ALGEBRA

Combinatorial Analysis

Harary, Frank. Note on an enumeration theorem of Davis and Slepian. Michigan Math. J. **3** (1955–1956), 149–153.

Let Φ be a set of b elements, called figures. A configuration of length s is a sequence of s figures. Let Γ be a permutation group of degree s and order h. Two configurations are Γ equivalent if there exists a permutation in Γ which sends one onto the other. B is the number of Γ-equivalence classes of configurations. A permutation in Γ, written as a product of disjoint cycles, is of type $(j)=(j_1, j_2, \cdots, j_s)$ if it has j_k cycles length k, for $k=1, 2, \cdots, s$. Let $h_{(j)}$ be the number of elements of Γ of type (j). A formula of Davis [Proc. Amer. Math. Soc. **4** (1953), 486–495; MR **14**, 1053] and Slepian [Canad. J. Math. **5** (1953), 185–193; MR **15**, 93] is

$$B=\frac{1}{h}\sum_{(j)} h_{(j)} b^{\Sigma j_k}.$$

Let each figure have associated with it an n-tuple of non-negative integers $(m_1, m_2, \cdots, m_n)$, the content of the figure, where n is the dimension of the content. The content of the configuration is the vector sum of the contents of the figures in it. In the following exposition of Pólya's formula only the case of dimension 2 is treated. Let $b_{m,n}$ be the number of figures of content (m, n) and let $b(x, y)=\sum_{m,n=0}^{\infty} b_{m,n}x^m y^n$ be the figure-counting series. Let $B_{m,n}$ be the number of Γ inequivalent configurations of content (m, n) and let $B(x, y)=\sum_{m,n=0}^{\infty} B_{m,n}x^m y^n$ be the configuration counting series. Let $f_1, \cdots, f_s$ be indeterminates. The cycle index of Γ is

$$Z(\Gamma)=\frac{1}{h}\sum_{(j)} h_{(j)} f_1^{j_1} f_2^{j_2}\cdots f_s^{j_s}.$$

For any power series $p(x, y)$ let $Z(\Gamma, p(x, y))$ be the series obtained from $Z(\Gamma)$ on replacing each indeterminate f_k by $p(x^k, y^k)$. Pólya's formula [Acta. Math. **68** (1937), 145–254] is $B(x, y)=Z(\Gamma, B(x, y))$.

The paper under review gives the above formulation to the Davis-Slepian formula, shows how it can be deduced from Pólya's formula, and gives some illustrations. As is indicated, Pólya's formula gives the number of Γ inequivalent configurations with a given content, while the Slepian-Davis formula gives more conveniently the number of Γ-inequivalent configuration without regard to content. *S. Sherman* (Philadelphia, Pa.).

Aigner, Alexander. Anordnungen mit der Rangkorrelation Null. Arch. Math. **7** (1956), 346–348.

Dykstra, Otto, Jr. A note on the rank analysis of incomplete block designs-applications beyond the scope of existing tables. Biometrics **12** (1956), 301–306.

The model is that of Bradley and Terry [Biometrika **39** (1952), 324–345; **41** (1954), 502–537; MR **17**, 56, 57]. The problem considered is how to find suitable initial estimates of the Π_i. These are to be substituted in certain equations as the first step in an iterative procedure to solve for the maximum likelihood estimates of the Π_i. Formulae are given for such initial estimates.

W. S. Connor (Washington, D.C.).

See also: Rao, p. 683.

Elementary Algebra

Segre, Beniamino. Generalizzazione di un'identità aritmetica di Conforto. Matematiche, Catania **9**, 97–105 (1954).

The author proves the identities

$$\sum_{h=1}^{r}\left(2\cos\frac{2h\pi}{r}\right)^n=r\sum\binom{n}{k},$$

$$\sum_{h=1}^{r}\left(2\sin\frac{2h\pi}{r}\right)^n=r\sum(-1)^{k+[n/2]}\binom{n}{k},$$

the summations on the right being over integer values of k satisfying

$$0\leqq k\leqq n,\ 2k\equiv n \pmod r.$$

By giving particular values r, n various arithmetic identities and congruences are obtained. *P. Du Val.*

See also: Boccioni, p. 638; Bottema, p. 684.

Linear Algebra

Nehari, Zeev. On bounded bilinear forms. Ann. of Math. (2) **65** (1957), 153–162.

The author considers quadratic forms of the type

$$A(a, a)=\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}\alpha(\mu+\nu)a_\mu a_\nu,$$

where the a_μ are variables subject to $\sum_0^\infty |a_\mu|^2=1$. If subject to this condition we always have $|A(a, a)|\leqq M$,

then the form is said to be bounded and the least such number M is called the bound of the form.

The author now obtains the bound of any such form as

$$M = \inf_{|z|<1} \sup |\sum_{0}^{\infty} \alpha(\nu) z^{\nu} + q(z)|,$$

where $q(z)$ ranges over all functions which are regular in $|z|<1$ and satisfy $q(0)=0$. He shows how to obtain from this the bound $M=\pi$ in Hilberts' case $\alpha(\nu)=\nu^{-1}$, $\nu \geqq 1$. He also obtains the bound π cosec $\pi\lambda$ in the more general case $\alpha(\nu)=(\nu+\lambda)^{-1}$ by a different method, and obtains for the first time the essentially unique case of equality.

There are a number of slips. In formula (10) read $\mu+1$ for $\mu-1$. In Theorem Ia an e^{2it} seems to have got lost and on p. 161 l. 11— read $e^{i\alpha\lambda}$ for $e^{i\lambda\theta}$ on the right hand side.

W. K. Hayman (London).

Foster, D. M. E. Indefinite quadratic polynomials in n variables. Mathematika **3** (1956), 111–116.

This paper is a sharpening of a result of H. Blaney [J. London Math. Soc. **23** (1948), 153–160; MR **10**, 511] which is as follows: If $q(x_1, \cdots, x_n)$ is an indefinite quadratic form in n variables with real coefficients and determinant $\Delta_n \neq 0$, for any $\gamma \geqq 0$, there is a number $\Gamma=\Gamma(\gamma, n)$ such that the inequalities

$$(1) \qquad \gamma|\Delta_n|^{1/n} < q(x_1+\alpha_1, \cdots, x_n+\alpha_n) < \Gamma|\Delta_n|^{1/n}$$

are solvable in integers $x_1, \cdots, x_n$ for any real $\alpha_1, \cdots, \alpha_n$.

The author shows that the same conclusion may be drawn for $\Gamma(\gamma, n)=\gamma+C_n\gamma^{1/2^n}+C_n'$, where C_n, C_n' are suitable positive numbers depending only on n.

B. W. Jones (Boulder, Colo.).

Newman, Morris; and Taussky, Olga. Classes of positive definite unimodular circulants. Canad. J. Math. **9** (1957), 71–73.

The authors extend to circulant matrices, a matrix theorem of Minkowski [Ges. Abh., Bd. I, Teubner, Leipzig-Berlin, 1911, pp. 3–144] for which an eighth order matrix [see Mordell, J. Math. Pures Appl. (9) **17** (1938), 41–46] was (then and now) the critical case. They show that an integral circulant of order 8 is unimodular, symmetric, and positive definite if and only if it is a power of the one whose first row is (2, 1, 0, −1, −1, −1, 0, 1). Odd powers of such a circulant are not of the form AA' for integral A of order 8. *H. Cohn.*

Schneider, Hans. Characteristic polynomials. Canad. J. Math. **9** (1957), 60–67.

Let $\mathbf{a}$ be a square matrix over field F and V its vector space. The characteristic polynomial $\det(\lambda I-\mathbf{a})$ of $\mathbf{a}$ can also be described as follows: form a composition series of V, considered as $F[\mathbf{a}]$-module. The factor groups are indecomposable $F[\mathbf{a}]$-modules; the product of the minimum polynomials of $\mathbf{a}$ on these factor groups is the characteristic polynomial of $\mathbf{a}$. Schneider gives a different and more complicated form of this definition and uses it to prove some theorems on characteristic polynomials.

G. Whaples (Bloomington, Ind.).

Mendelsohn, N. S. Some properties of approximate inverses of matrices. Trans. Roy. Soc. Canada. Sect. III. (3) **50** (1956), 53–59.

The paper describes investigations into conditions in which an approximate inverse matrix B of A may be such that BA is a good approximation (B a "left" inverse), AB a relatively poor approximation (B a "right" inverse) to the unit matrix I. It may happen that a first correction $B_1=B(2I-AB)$ is such that $B_1A-I<BA-I$, while $AB_1-I>AB-I$ in every element, though this result does not extend to all subsequent corrections. The error in the solution of linear equations depends on the error in B regarded as a left inverse only. Connections are obtained between these results and various measures of condition of A. *L. Fox* (Berkeley, Calif.).

See also: Marcus and Thompson, p. 634; Butson, p. 636; Hodges, p. 643; Basotti, p. 666; Schwarz, p. 676; Thompson, p. 683; Abzug, p. 684.

Polynomials

Bottema, O. On the roots of the biquadratic equation. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 407–410.

In connection with the fourth degree equation

$$x^4+ax^2+bx+c=0$$

with real coefficients and discriminant $D \neq 0$, there are three cases: (I) four roots all real, (II) two pairs of conjugate complex roots, (III) two roots real and a pair of conjugate complex roots. Kronecker has studied these three cases geometrically in three dimensional space using the fact that the roots are functions of the three variables a, b, c. The author remarks that D may be expressed in the form a^6K, where K is a function of $u=12ca^{-2}$ and $v=(27/2)b^2a^{-3}$. Thus a geometrical study of the regions in the uv-plane into which the whole plane is divided by the equation $D=0$ appears to be appropriate. Such a two dimensional picture does indeed suffice to decide which of the three cases I, II, or III prevails as long as $D \neq 0$. The method also works in the limiting cases $D=0$, except when $v=0$ and $0<u<3$. *D. C. Lewis* (Baltimore, Md.).

Marcus, M.; and Thompson, R. A note on symmetric functions of eigenvalues. Duke Math. J. **24** (1957), 43–45.

Let A be a non-negative Hermitian linear mapping in n-dimensional unitary space, with eigenvalues

$$\rho_1 \leqq \rho_2 \leqq \cdots \leqq \rho_n.$$

Let $E_r(a_1, \cdots, a_k)$ be the elementary symmetric function of order r in $a_1, \cdots, a_k$, and let $0 \leqq s \leqq 1$. Then

$$\min_R \max_x E_r[(Ax_1, x_1)^s, \cdots, (Ax_k, x_k)^s] = \binom{k}{r}\left(\frac{\rho_{p+1}+\cdots+\rho_{p+k}}{k}\right)^{sr},$$

the maximum being taken over all orthonormal systems $x_1, \cdots, x_k$ with k elements in a fixed $(p+k)$-dimensional vector subspace R_{p+k}, and the minimum over all such subspaces. Generalisations of this result from elementary symmetric functions to Schur-convex and Schur-concave functions [A. Ostrowski, J. Math. Pures Appl. (9) **31** (1952), 253–292; MR **14**, 625] are indicated.

A similar result is proved for expressions of the form

$$\sum_{\sigma \in S} c_\sigma \rho_\sigma,$$

where S is a set of p integers in $[1, n]$, and $c_\sigma \geqq 0$, A now being a general Hermitian mapping. *F. Smithies.*

Seres, I. Lösung und Verallgemeinerung eines Schurschen Irreduzibilitätsproblems für Polynome. Acta Math. Acad. Sci. Hungar. **7** (1956), 151–157. (Russian summary)

I. Schur proposed as a problem to prove that if $a_1, \cdots, a_n$ are distinct rational integers, then the polynomial $f(x)=(x-a_1)\cdots(x-a_n)-1$ is irreducible in the field of rationals. This was proved and generalized by W. Flügel. Further generalizations were proved by A. Brauer, R. Brauer, and H. Hopf. In this paper the author proves three other generalizations as follows. I. If $P(x)$ is a polynomial with leading coefficient unity, whose zeros are distinct rational integers, then the polynomial

$$F(x)=[P(x)]^2+1$$

is irreducible in the field of rationals. II. If $f_m(x)$ is the monic polynomial whose zeros are the primitive mth roots of unity, and $P(x)$ is a monic polynomial of degree greater than 4 whose zeros are distinct rational integers, then $f_m(P(x))$ is irreducible in the field of rationals. If the degree of $P(x)$ is less than 5, the same result holds with some exceptions. III. Let $P(x)$ be a monic polynomial of degree greater than 5 whose zeros are distinct rational integers, let $Q(x)$ be a monic polynomial with rational integral coefficients and of degree less than that of $P(x)$, let $R(x)=P(x)Q(x)$, and let $f_m(x)$ be the monic polynomial whose zeros are the primitive mth roots of unity; then $f_m(R(x))$ is irreducible in the field of rationals.

O. Frink (University Park, Pa.).

See also: Thompson, p. 683.

Continued Fractions

Wall, H. S. Partially bounded continued fractions. Proc. Amer. Math. Soc. **7** (1956), 1090–1093.

The continued fraction (1) $K(a_n/1)$ is called partially bounded if a subsequence of the sequence $\{a_n\}$ of its elements is bounded. The author proves that if (1) is partially bounded, and if in addition the sequences of even and odd approximants are convergent, then (1) converges. This together with results from the theory of positive definite continued fractions is used to prove the following twin convergence region criterion: Let $0<r\leqq 1$, and $0<\varepsilon$, then (1) converges if for all $n\geqq 1$

$$|a_{2n-1}|\leqq r^2|a_{2n}|\geqq(1+r)^2+\varepsilon.$$

The author states that for $r<1$ this theorem is due to Thron [Duke Math. J. **10** (1943), 677–685; MR **5**, 118]. For $r=1$ the result had also been previously established by Cowling, Leighton, and Thron [Bull. Amer. Math. Soc. **50** (1944), 351–357; MR **6**, 47]. The paper concludes with a reference to an unpublished dissertation [Univ. of Texas, 1950] by G. Copp in which somewhat better twin convergence regions were obtained. For $r<1$ all of these criteria are contained in a recent result of V. Singh and Thron [Proc. Amer. Math. Soc. **7** (1935), 277–282; MR **17**, 1076]. The continued fraction $K(c_n^2/1)$ converges if $|c_{2n-1}|\leqq r$ and $|c_{2n}\pm i|\geqq r$.

W. J. Thron.

Wright, Fred M. Some sufficient conditions for a determinate Hamburger moment sequence. Proc. Amer. Math. Soc. **7** (1956), 1040–1043.

The author shows that each of the conditions

$$\sum_{n=0}^{\infty}\left(\frac{\mu_{2n}}{\mu_{2n}\mu_{2n+2j}-\mu_{2n+j}^2}\right)^{1/(2n+2j)}=\infty,$$

where j may be an arbitrary positive integer, is sufficient for the sequence $\{\mu_n\}$ to be a determinate Hamburger moment sequence. The proof is based on a sufficient condition, due to R. Nevalinna. This condition is stated in terms of the sequence $\{a_n\}$ of the J fraction expansion $K_{n=0}^{\infty}(a_n|(z+b_{n+1}))$ associated with the moment sequence.

W. J. Thron (Boulder, Colo.).

Smith, G. S. Expression of irrationals of any degree as regular continued fractions with integral components. Amer. Math. Monthly **64** (1957), 86–88.

The author applies Thiele's well-known theorem for the development of $f(x+h)$ into a continued fraction to the function $f(x)=x^{1/m}$ (m an integer or a rational). In this manner he obtains in a formal way continued fractions for the mth root of a positive integer N. Numerical example:

$2^{2/3}=$

$$1+\frac{2|}{|3}+\frac{1|}{|2}+\frac{5|}{|9}+\frac{4|}{|2}+\frac{8|}{|15}+\frac{7|}{|2}+\frac{11|}{|21}+\frac{10|}{|2}+\cdots.$$

J. Popken (Amsterdam).

Djerasimović, Božidar. Über die Kettenbruchentwicklung quadratischer Irrationalzahlen. Math. Z. **66** (1956), 228–239.

In a previous paper [Math. Z. **62** (1955), 320–239; MR **17**, 255] the author had introduced a set of symbols to express certain operations on ordered n-triples of numbers to facilitate the derivation of results in the theory of regular continued fractions. In this paper this program is continued and applications are made to the behavior of regular continued fraction expansions of two conjugate quadratic irrational numbers.

W. J. Thron.

See also: Lochs, p. 643; Berg, p. 645; Schwarz, p. 676.

Partial Order Structures

Wolk, E. S. Some representation theorems for partially ordered sets. Proc. Amer. Math. Soc. **7** (1956), 589–594.

Using the definition of ideal and dual ideal in a partially ordered set proposed by the reviewer [Amer. Math. Monthly **61** (1954), 223–234; MR **15**, 848], the author proves some representation theorems for partially ordered sets generalizing similar theorems for lattices. He also generalizes some results of H. Wallman connecting the topology of a T_1 space with the properties of a basis for the closed sets of the space, namely to the case where the basis is not a lattice but merely a partially ordered set.

The author first define the notions of finite intersection property, disjunctive, and covering family in a partially ordered set, and then derives a necessary and sufficient condition that a partially ordered set be isomorphic to the Boolean algebra 2^S of all subsets of some finite or infinite set S. He obtains as a corollary the theorem of G. Birkhoff and M. Ward that a complete atomic lattice with unique complements is isomorphic to 2^S.

He then defines the concepts of reducible and prime dual ideal of a partially ordered set, and shows that a disjunctive partially ordered set in which each maximal dual ideal is prime, is isomorphic to a basis for the closed sets of some compact T_1 space. He next gives a characterization of the lattice of all closed subsets of a compact T_1 space. Finally,

generalizing a result of H. Wallman, he shows that if B is any admissible basis for the closed subsets of a compact T_1 space, then the structure of B as a partially ordered set determines the space to within a homeomorphism. These results suggest that the definition of ideal which is used is a convenient one. *O. Frink* (University Park, Pa.).

Novotný, Miroslav. Bemerkung über die Darstellung teilweise geordneter Mengen. Publ. Fac. Sci. Univ. Masaryk **1955**, 451–458. (Czech and Russian summaries)

The author considers three representations (involving either sequences of zeros and ones, or subsets of a set) of a partially ordered set G, with each of which there is associated a cardinal number. The smallest cardinal number for which such a representation of G exists is shown to be the same for all three, and, in case G is ordered and infinite, coincides with the smallest cardinal number $\mathfrak{d}$ such that there is a subset D of G of power $\mathfrak{d}$ which is dense in G. *F. Bagemihl* (Notre Dame, Ind.).

Butson, A. T. Matrices with elements in a Boolean ring. Canad. J. Math. **9** (1957), 47–59.

Let $\mathfrak{B}$ be a Boolean ring with unit and let $\mathfrak{B}_n$ be the ring of all n by n matrices with elements in $\mathfrak{B}$. The author shows that $\mathfrak{B}$ is an Hermite ring [I. Kaplansky, Trans. Amer. Math. Soc. **66** (1949), 464–491; MR **11**, 155] by explicit reduction of each $A \in \mathfrak{B}_n$ to Hermite canonical form $H=UA$, where U is a product of elementary matrices. A second, more complicated, canonical form is also given. Some usual consequences of Kaplansky's observation that R_n is an Hermite ring whenever R is [op. cit., p. 469] are then derived. {Reviewer's comments: The major result of this paper was noted but was relegated to minor remarks by Kaplansky [op. cit., p. 468, Remark 2, and p. 474]. For a neat characterization of commutative Hermite rings, see L. Gillman and M. Henriksen [Trans. Amer. Math. Soc. **82** (1956), 362–365; MR **18**, 9].}

M. F. Smiley (Iowa City, Iowa).

See also: Borovskiĭ, p. 664.

Rings, Fields, Algebras

Scott, Dana. Equationally complete extensions of finite algebras. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 35–38.

"The purpose of this note [is to prove that] the set of identities of a finite algebra has only finitely many equationally complete [e.c.] extensions." An effective test for finite A to be e.c. is given. For A an abstract algebra, of fixed species, let $I(A)$ be the set of its identities, and, for any fixed $n \geqq 2$, let $P(I)$ be the free algebra on n generators defined by $I(A)$. A is e.c. if $I(A)$ is maximal. Since $I(A) \subseteq I(P)$, A will be e.c. if and only if $P(I)$ is e.c. and $I(A)=I(P)$. If A is finite, $P(I)$ is finite, with only finitely many homomorphs, in particular homomorphs of the form $P(J)$ for maximal J containing I.

R. C. Lyndon (Berkeley, Calif.).

Tarski, Alfred. Equationally complete rings and relation algebras. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 39–46.

All equationally complete [e.c.; see the preceding review] associative rings and relation algebras are determined. In the case at hand, every e.c. class is determined by a non-trivial algebra without proper non-trivial subalgebras or homomorphs. A minimal simple associative ring is shown to be either a field, or a zero-ring, of prime order. The corresponding identities are $ab=ba$, $pa=0$, and either $a^p=a$ or $ab=0$. A non-trivial ring is e.c. if and only if it is a p-ring or a p-zero-ring for some prime p. The minimal simple relation algebras contain only the constants 0, 1, 1′, 0′ (the empty, universal, identity, and diversity relations), where the product 0′; 0′ is one of 0, 1, 1′. This determines three classes K_1, K_2, K_3, and each K_i is defined by an equation giving the value of the product 0′; 0′, together with an equation expressing that every element is equal to one of 0, 1, 0′, 1′. *R. C. Lyndon* (Berkeley, Calif.).

Fujiwara, Tsuyoshi. Note on free algebraic systems. Proc. Japan Acad. **32** (1956), 662–664.

The author extends certain results by Shoda [Osaka Math. J. **1** (1949), 182–225; MR **11**, 308] on the existence of free algebraic systems defined for an arbitrary family of one-valued operations with a given set of relations.

O. Ore (New Haven, Conn.).

Tominaga, Hisao. Galois theory of simple rings. Math. J. Okayama Univ. **6** (1956), 29–48.

L'auteur ne considère que des anneaux simples satisfaisant à la condition minimale et ayant même élément unité. Si $R \supset S$ sont deux tels anneaux, R est dit galoisien sur S si le commutant S' de S dans R est simple et si S est l'anneau des invariants d'un groupe d'automorphismes de R. La théorie de Galois pour un tel couple d'anneaux vise à établir une correspondance biunivoque entre certains sous-anneaux simples $S \subset T \subset R$ et certains sous-groupes du groupe $\mathfrak{G}$ des S-automorphismes de R, de la façon habituelle. Lorsque $[R:S]$ est fini, ce programme a été rempli par T. Nakayama [Trans. Amer. Math. Soc. **73** (1952), 276–292; MR **14**, 240]; l'auteur aborde le cas où $[R:S]$ est infini, mais conserve de très fortes restrictions de finitude: pour toute partie finie F de R, il suppose qu'il existe un anneau simple $N \subset R$, contenant $S \cup F$, globalement invariant par $\mathfrak{G}$, tel que $[N:S] < +\infty$ et que le bicommutant N'' de N dans R soit simple; en outre, il suppose S' de rang fini sur le centre de R. Moyennant quoi, il peut montrer que le groupe $\mathfrak{G}$, topologisé par la méthode de Krull, est localement compact; la correspondance galoisienne a lieu entre les anneaux simples $S \subset T \subset R$ dont le commutant T' dans R est simple et les sous-groupes de $\mathfrak{G}$ laissant ces anneaux invariants, sous-groupes que l'auteur ne caractérise d'ailleurs pas de façon intrinsèque. Il y a aussi un théorème de prolongement d'isomorphisme, du type habituel. *J. Dieudonné.*

Moriya, Mikao. Zusammenhang zwischen Derivationsmodul und 2-Kohomologiegruppe. II. Math. J. Okayama Univ. **6** (1956), 49–69.

This completes the proof of the isomorphism

$$H^1(R/r, R/P^m) \approx H^2(R/r, R/P^m)$$

of the cohomology groups associated with a finite separable algebraic extension of a field that is complete for a discrete rank 1 valuation. The isomorphism was established for sufficiently large m in the first part of this work [J. Math. Soc. Japan **7** (1955), 444–452; MR **18**, 8] and is proved here to hold for all $m \geqq 0$. The notation used above is explained in the cited review.

G. P. Hochschild (Princeton, N.J.).

Northcott, D. G. Semi-regular local rings. Mathematika 3 (1956), 117–126.

Un anneau local Q est dit semi régulier si, pour tout système de paramètres $(x_1, \cdots, x_d)$, l'idéal maximal $\mathfrak{m}$ n'est pas un idéal premier associé de $(x_1, \cdots, x_{d-1})$; alors $Q/(x_1, \cdots, x_r)$ est semi régulier pour tout $j \leq d$, et $(x_1, \cdots, x_j)$ est équidimensionnel. En se servant d'un lemme de C. Lech (pour tout système de paramètres $(x_1, \cdots, x_d)$ la multiplicité $e(x_1, \cdots, x_d)$ est égale à $\lim_{\max(n_i(i))\to\infty} L(x_1^{n(1)}, \cdots, x_d^{n(d)})/n(1)\cdots n(d))$ on montre que, dans un anneau local semi régulier, la longueur $L(x_1, \cdots, x_d)$ est égale à la multiplicité $e(x_1, \cdots, x_d)$. Réciproquement s'il existe, dans un anneau local Q, un système de paramètres engendrant un idéal dont la longueur est égale à la multiplicité, alors Q est semi régulier. Le complété d'un anneau local semi régulier est semi régulier. Si Q est semi-régulier, $Q_{\mathfrak{p}}$ est semi régulier pour tout idéal premier $\mathfrak{p}$. Pour tout idéal $\mathfrak{a}$ d'un anneau local semi régulier, on a $\dim(Q)=\dim(Q/\mathfrak{a})+\operatorname{rang}(\mathfrak{a})$; si deux idéaux premiers $\mathfrak{p}$ et $\mathfrak{p}'$ de Q sont tels que $\mathfrak{p}\subset\mathfrak{p}'$ et qu'il n'existe aucune idéal premier entre $\mathfrak{p}$ et $\mathfrak{p}'$, on a

$$\dim(Q/\mathfrak{p})=1+\dim(Q/\mathfrak{p}').$$

La classe des anneaux locaux semi réguliers a été étudiée independemment par M. Nagata et D. Rees dans les oeuvres analysées ci-dessous. *P. Samuel.*

★**Nagata, Masayoshi. The theory of multiplicity in general local rings.** Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 191–226. Science Council of Japan, Tokyo, 1956.

L'auteur donne un exposé d'ensemble de la théorie des multiplicités dans les anneaux locaux, et y ajoute un grand nombre de résultats nouveaux et d'améliorations de résultats connus. Définition des multiplicités dans les modules sur les anneaux semi locaux. Formules pour le passage d'un anneau local $\mathfrak{o}$ à un anneau semi local $\mathfrak{o}'$ qui est un $\mathfrak{o}$-module de type fini. Si $\mathfrak{o}$ est un anneau local, $\mathfrak{p}_i$ les idéaux premiers de (0) tels que $\dim(\mathfrak{o}/\mathfrak{p}_i)=\dim(\mathfrak{o})$, $\mathfrak{q}_i$ les composantes primaires de (0) correspondantes, et $\mathfrak{q}$ un idéal primaire pour l'idéal maximal, on a $e(\mathfrak{q})=\sum_i e((\mathfrak{q}+\mathfrak{q}_i)/\mathfrak{q}_i)$ (e désignant la multiplicité d'un idéal). Si (0) est primaire et si $\mathfrak{p}$ est son radical, on a $e(\mathfrak{q})=e((\mathfrak{q}+\mathfrak{p})/\mathfrak{p})$ (longueur $\mathfrak{o}_{\mathfrak{p}}$). On en déduit une formule d'associativité valable pour les anneaux locaux quelconques. Pour qu'un anneau local soit régulier, il faut et il suffit qu'il soit équidimensionnel et de multiplicité 1. Si $\mathfrak{p}$ est un idéal premier d'un anneau local $\mathfrak{o}$, si

$$\dim(\mathfrak{o})=\dim(\mathfrak{o}/\mathfrak{p})+\dim(\mathfrak{o}_{\mathfrak{p}}),$$

et si $\mathfrak{p}$ est analytiquement non ramifié, la multiplicité de $\mathfrak{o}_{\mathfrak{p}}$ est au plus égale à celle de $\mathfrak{o}$. Etude des anneaux locaux dans lesquels il existe un système de paramètres engendrant un idéal dont la multiplicité est égale à la longueur [cf. l'analyse ci-dessus]. *P. Samuel.*

Rees, D. The grade of an ideal or module. Proc. Cambridge Philos. Soc. **53** (1957), 28–42.

L'auteur décrit d'intéressantes applications de l'Algèbre Homologique à la théorie des idéaux. Soit A un anneau noethérien. On appelle "grade" d'un A-module M le plus petit entier k tel que $\operatorname{Ext}_A^k(M, A)\neq 0$; le grade d'un idéal $\mathfrak{a}$ est par définition celui de $A/\mathfrak{a}$. Si M est un A-module de grade k et N un A-module de dimension homologique $j<k$, on a $\operatorname{Ext}_A^p(M, N)=0$ pour $p<k-j$; en particulier, si $\mathfrak{a}$ est un idéal de grade k et $\mathfrak{b}$ un idéal de dimension homologique $j<k$, on a $\mathfrak{b}:\mathfrak{a}=\mathfrak{b}$, et tout idéal premier associé de $\mathfrak{b}$ a un grade $\leq j$. Un idéal $\mathfrak{g}=(g_1, \cdots, g_k)$ de A est dit général si l'on a

$$(g_1, \cdots, g_{i-1}):g_i=(g_1, \cdots, g_{i-1})$$

pour $i=1, \cdots, k$. Un module (ou idéal) est dit parfait si sa dimension homologique coïncide avec son grade; cette notion coïncide avec la notion classique d'idéal parfait. Etude de l'anneau gradué $F(\mathfrak{g})=\sum_n \mathfrak{g}^n/\mathfrak{g}^{n+1}$ associé à un idéal général $\mathfrak{g}$; c'est un anneau de polynômes sur $A/\mathfrak{g}$, et la réciproque est vraie si A est local; tous les idéaux premiers associés aux puissances de $\mathfrak{g}$ ont le même grade. Etude des anneaux (dits de type U) tels que grade $\mathfrak{a}\geq$ hauteur $\mathfrak{a}$ pour tout idéal $\mathfrak{a}$; pour que A soit de type U il faut et il suffit que tout idéal général soit équidimensionnel; en particulier les anneaux de polynômes sur un corps et les anneaux locaux réguliers sont de type U d'après les th. de Macaulay et Cohen. Dans un anneau de type U toute puissance d'un idéal général et tout idéal parfait sont équidimensionnels. Caractère local de la notion d'anneau de type U. Si, dans un anneau local A, l'idéal maximal $\mathfrak{m}$ a un grade égal à sa hauteur ($=\dim(A)$), alors A est de type U. Pour qu'un anneau local A soit de type U, il faut et il suffit que tout idéal $\mathfrak{q}$ engendré par un système de paramètres soit général, ou encore que la multiplicité de $\mathfrak{q}$ soit égale à sa longueur [des résultats analogues ont été trouvés indépendemment par M. Nagata dans l'oeuvre analysée ci-dessus]. Pour qu'un quotient $A/\mathfrak{a}$ d'un anneau local régulier A soit de type U, il faut et il suffit que $\mathfrak{a}$ soit un idéal parfait. *P. Samuel.*

Yoshida, Michio. Some remarks on Noetherian rings. Canad. J. Math. **9** (1957), 35–37.

Utilisation du „lemme d'Artin-Rees" [Rees, Proc. Cambridge Philos. Soc. **52** (1956), 155–157; MR **17**, 576] pour donner des conditions suffisantes de validité de la formule $\mathfrak{a}^{n+k}:\mathfrak{a}^k=\mathfrak{a}^n$ ($\mathfrak{a}$ idéal d'un anneau noethérien, n grand) et de formules voisines. *P. Samuel.*

Steinfeld, O. Über die Quasiideale von Ringen. Acta Sci. Math. Szeged **17** (1956), 170–180.

In this paper the author continues his investigations [Acta Math. Acad. Sci. Hungar. **6** (1955), 479–484; MR **17**, 1180] on the quasiideals of associative rings. The author's concept of the quasiideal is defined as follows: a submodule A of the associative ring R is called a quasi-ideal of R if $RA\cap AR\subseteq A$. We should like to point out the following two theorems which constitute the main results of the paper: (1) A minimal quasiideal A of a ring R is either a zero-ring or a skewfield. This latter must have the form eRe, where e is the unit element of the skewfield A. (2) Every minimal quasiideal of a radical-free ring (i.e. a ring without non-zero nilpotent left ideals) is an intersection of some minimal left ideal with some minimal right ideal of the ring. *A. Kertész* (Debrecen).

Lesieur, Léonce; et Croisot, Robert. Sur la décomposition en idéaux primaires dans un anneau non nécessairement commutatif. C. R. Acad. Sci. Paris **243** (1956), 1988–1991.

Let R be a ring, not necessarily commutative, that satisfies either the ascending or descending chain condition for left ideals. An ideal Q is (right) primary if the relation $Q\cdot B\neq Q$, where B is a two-sided ideal, implies $B^n\subseteq Q$ for some positive integer n. Here $Q\cdot B$ is the right quotient ideal of Q by B, and Q may be a two-sided ideal or merely a left ideal. Two generalizations of a primary

ideal are proposed: A (left or two-sided) ideal Q is (right) secondary if the relation $Q\cdot B\neq Q$, where B is a two-sided ideal, implies the existence of two-sided ideals $L_1, L_2, \cdots, L_n$ and positive integers $k_0, k_1, \cdots, k_n$ such that $B^{k_0}L_1B^{k_1}L_2\cdots L_nB^{k_n}\subseteq Q$, and $Q\cdot L_j=Q$ $(j=1, 2, \cdots, n)$. A two-sided ideal is (right) tertiary if, when B and X are two-sided ideals, the relations $Q\cdot B\neq Q$ and $(Q\cdot B)\cap X=Q$ imply $X=Q$. A left ideal is tertiary if the above property holds for any two-sided ideal B and any left ideal X. The following results are announced, proofs to be found in a forthcoming publication. All primary ideals are secondary and all secondary ideals are tertiary. If R is commutative all tertiary ideals are primary. Every tertiary ideal Q (left or two-sided) has a maximum left residual which is a prime ideal P, the associated prime of Q. Every ideal in R is an irredundant intersection of a finite number of tertiary ideals with distinct associated primes. In any two such representations of the same ideal, the number of tertiary components is the same and the two sets of associated primes are the same. If the descending chain condition for left ideals holds in R then the notions of primal, secondary, and tertiary ideals are equivalent, and for an ideal to be tertiary it is necessary and sufficient that it have a unique left residual which is a proper prime ideal. *D. C. Murdoch* (Vancouver, B.C.).

Ghika, Al. Algèbres corpoïdales. Com. Acad. R. P. Romîne **6** (1956), 239–243. (Romanian. Russian and French summaries)

An algebra A with unity element over a skew field K is called field-like (corpoidală) if the group G of its invertible elements contains a Hamel basis of the left vector space A over K [see Ghika, same Com. **5** (1955), 317–323, quoted as M, where K is the field of real numbers; MR **17**, 767]. Examples are given (e.g., field extensions, complex valued functions over a set) and there are several propositions one of which is false: viz., if H is a normal subgroup of G then G contains a subgroup M isomorphic with G/H (take $A=R(2^{1/3})$ over the field R of rational numbers; G is the multiplicative group of A and let H be the multiplicative group of R). A last section deals with paranormes (see M) of vector spaces over field-like algebras. *G. K. Kalisch* (Minneapolis, Minn.).

Boccioni, Domenico. Q-pseudogruppi complementarizzabili. Rend. Sem. Mat. Univ. Padova **26** (1956), 85–123.

The author assumes that two pseudo-groups (multiplicative systems) Q and G are given such that Q acts as an operator domain on G. A product $a\cdot u$ $(a\in Q,\ u\in G)$ obeys the rules $a\cdot u\in G$, $a(u+u_1)=a\cdot u+a\cdot u_1$, $a(a_1\cdot u)=(aa_1)\cdot u$. Operations in G are considered additive, those in Q multiplicative, neither need be commutative. The author gives certain simple conditions under which G can be extended to a quotient system with zero and negative elements such that linear equations $\alpha\xi=\xi_1$ have unique solutions for the so-called simplifiable elements α in Q. The results may be applied to rings and other systems and contain as special cases theorems obtained by Asano, Murata and the reviewer. *O. Ore* (New Haven, Conn.).

Patterson, E. M. Linear algebras of genus one admitting a unit element. J. London Math. Soc. **32** (1957), 88–94.

M. S. Knebelman [Ann. of Math. (2) **36** (1935), 46–56] has defined the nullity of an n-dimensional linear algebra A to be the least number of linearly independent elements which generate A, in the sense that every element of A is a linear combination of products of these elements. Knebelman also defined the genus of A to be the difference between its dimension and its nullity. In a previous paper [J. London Math. Soc. **31** (1956), 326–331; MR **18**, 109] the author has determined all algebras of dimension n and genus zero. In this paper he obtains the following results, where F is any field containing at least four distinct elements. (I) Let A be an associative algebra over F of dimension $n(\geqq 3)$, genus one, and having a unit element. Then A is isomorphic with either (i) the algebra G_n defined by $e_1e_p=e_pe_1=e_p$ $(p=1, \cdots, n)$, $e_pe_q=0$ $(p, q=2, \cdots, n)$ or (ii) the algebra H_n defined by $e_1e_p=e_p=e_pe_1$ $(p=1, \cdots, n)$, $e_pe_q=e_p$ $(p, q=2, \cdots, n)$. (II) Let B be an algebra over F of dimension $n(>1)$ and genus zero such that B is neither a Lie algebra nor an associative algebra. Then both the left and right multiplication algebras of B are isomorphic with H_{n+1}. (III) Let A be a commutative algebra over the complex field, of dimension $n(\geqq 3)$, genus one and having a unit element. Then A is isomorphic with G_n or with one of the algebras $J_n(r)$ $(r=1, \cdots, n-1)$ defined by $e_1e_p=e_pe_1=e_p$ $(p=1, \cdots, n)$, $e_pe_q=\delta_{pq}e_1$ $(p, q=2, \cdots, r+1)$, $e_pe_q=0$ otherwise, where δ_{pq} is the Kronecker delta. *R. L. San Soucie* (Eugene, Ore.).

Seligman, George B. Characteristic ideals and the structure of Lie algebras. Proc. Amer. Math. Soc. **8** (1957), 159–164.

Let L be a finite-dimensional Lie algebra over a field F. An ideal I of L is called a characteristic ideal in L if $ID\subseteq I$ for every derivation D of L. The (unique) maximal solvable characteristic ideal in L is called the characteristic radical, or c-radical, of L. L is called characteristically semi-simple (css.) if its c-radical is (0); characteristically simple (c-simple) if its only characteristic ideals are L and (0) and if $[LL]=L$; completely semi-simple (comp ss.) if L can be written as a direct sum of a set of ideals in L which are characteristic simple algebras. The main results of the paper follow: (1) If R is the c-radical of L, then L/R is css. (2) Every proper ideal in a c-simple algebra is nilpotent. (3) Every comp ss. Lie algebra is css. (4) A css. L contains a non-zero comp ss. characteristic ideal. (5) Every css. L contains a unique non-zero maximal comp ss. characteristic ideal S. The annihilator of S is (0) and if we denote by $\mathrm{ad}_S(x)$ the restriction to S of the mapping $\mathrm{ad}(x)$, for x in L, it follows that x into $\mathrm{ad}_S(x)$ is an isomorphism of L into the derivation algebra of S. Finally, the author observes that, in the characteristic zero case, the theory here outlined collapses to the classical case. He gives an example to show that this need not occur in the case of prime characteristic. *R. L. San Soucie.*

Strauss, Martin. Über eine nicht-archimedische Addition und die Frage ihrer Verwendung in der Physik. Wiss. Z. Humboldt-Univ. Berlin. Math.-Nat. Reihe **5** (1955/56), 93–97. (Russian, English and French summaries)

Einstein's law of addition of velocities suggests to the author the introduction of the binary operation

$$a\dot{+}b=(a+b)(1+ab)^{-1};$$

on the other hand Gramatzki [Astr. Nachr. **234** (1929), 17–32] had proposed another binary operation, viz. $a\hat{+}b=a+b-ab$. The paper discusses formal properties of these two operations, shows that one cannot be reduced to the other by a transformation, and points out the advantage of the first operation when it comes to applications. *G. Y. Rainich* (Providence, R.I.).

Linnik, Yu. V. A problem in differential algebra originating in mathematical statistics. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 3(69), 169–170. (Russian)

The background for the problem is a previous paper of the author [Vestnik Leningrad. Univ. **11** (1956), no. 1, 35–48; MR **17**, 983]. In the problem an infinite set of algebraic differential equations is given and one asks whether they have as consequence an equation of the second order. It is noted that Ritt's basis theorem shows that the problem is one of elimination on a suitable finite subset of the equations. *I. Kaplansky.*

Hilton, P. J. Remark on the tensor product of modules. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 325–328.

Soit Λ un anneau avec unité. Un Λ-module à droite A possède la propriété (P) si, pour tout monomorphisme $B \to C$ de Λ-modules à gauche, l'application

$$A \otimes_\Lambda B \to A \otimes_\Lambda C$$

a un noyau nul. Cela revient à dire que $\mathrm{Tor}_1^\Lambda(A, X)=0$ pour tout Λ-module à gauche X [cf. Cartan et Eilenberg, Homological algebra, Princeton, 1956; MR **17**, 1040]. Puisque Tor commute avec les limites directes, toute limite directe de modules ayant la propriété (P) possède cette propriété; donc si tout sous-module de type fini de A est projectif, A possède la propriété (P). Supposons désormais que Λ soit sans diviseur de zéro; si A possède (P), A est sans torsion; la réciproque est vraie quand les idéaux à droite et les idéaux à gauche de Λ sont principaux, car alors tout module de type fini et sans torsion est libre. *H. Cartan* (Paris).

See also: Tarski, p. 632; Scott, p. 639; Suprunenko, p. 639; McCarthy, p. 641; Masuda, p. 643; Iwasawa, p. 643; Kubota, p. 643; Tannaka, p. 644; Terada, p. 644; Schieferdecker, p. 658; Dixmier and Lister, p. 659; Gurevič, p. 659; Lyndon, p. 659; Rudin, p. 661; Borovskiĭ, p. 664; Mendelsohn, p. 665; Zaddach, p. 665; Klingenberg, p. 667.

Groups, Generalized Groups

★**Sielaff, Klaus. Einführung in die Theorie der Gruppen.** Schriftenreihe zur Mathematik, Heft 4. Otto Salle Verlag, Frankfurt am Main-Hamburg, 1956. ii+82 pp. DM 5.80.

This elementary introduction to the theory of groups is intended for pupils in their last years at school and mathematically interested laymen. In the first part (17 pp.) the group concept is introduced not formally, but by way of examples of groups of numbers and groups of Euclidean plane motions. The second part (42 pp.) develops the group concept axiomatically and ranges in the customary way over the topics of subgroups, cosets, isomorphism, homomorphism, conjugacy, normal subgroups and factor groups. The examples are chosen mainly from geometry and are carefully set out both as motivation and illustration. The third part (22 pp.) treats the classes of cyclic groups and dihedral groups in much detail; the complete subgroup diagrams of the D_6 and D_{15} are given. Finally the groups of the five Platonic solids are briefly dealt with. Easy exercises are interspersed in the text.

K. A. Hirsch (London).

Baer, Reinhold. Noethersche Gruppen. Math. Z. **66** (1956), 269–288.

A Noetherian group is one which satisfies the ascending chain condition for subgroups, or, equivalently, all of whose subgroups are finitely generated. This is, of course, much stronger than assuming the group itself to be finitely generated; how much stronger is hard to say, since all known Noetherian groups are finite extensions of soluble groups (with finite derived lengths). The aim of the paper under review is to give conditions under which a finitely generated group is necessarily Noetherian. It is characteristic of these conditions that they permit the construction of an ascending series of Noetherian normal subgroups, so that nothing is said, for instance, about the possibility of infinite simple Noetherian groups. The simplest criterion states that a finitely generated group is Noetherian if every homomorphic image (not 1) has a normal subgroup (not 1) which is either finite or is a finitely generated abelian group. It appears not to be known whether it is sufficient to assume a Noetherian normal subgroup. *Graham Higman* (Oxford).

Černikova, N. V. Groups with complemented subgroups. Mat. Sb. N.S. **39(81)** (1956), 273–292. (Russian)

The author (née Baeva) supplies proofs of results announced in her earlier paper [Dokl. Akad. Nauk SSSR (N.S.) **92** (1953), 877–880; MR **15**, 503; see also S. N. Černikov, ibid. **92** (1953), 891–894; Mat. Sb. N.S. **35(77)** (1954), 93–128; MR **15**, 504; **16**, 565]. Using the fact that a completely factorizable group G is a splitting extension of one completely factorizable abelian group A by another such B with further conditions on A, she shows how to obtain a standard set of generators for G. Such a G is characterized by its appropriate imbedding in a strong direct sum of certain cyclic groups and/or certain splitting extensions of cyclic groups. A strong direct sum of an infinite number of copies of a non-abelian group of order 6 is shown not to be completely factorizable even though it is a splitting extension of an abelian completely factorizable group by another such abelian group. *F. Haimo.*

Kemhadze, Š. S. Regular p-groups without elements of infinite height. Soobšč. Akad. Nauk Gruzin. SSR **17** (1956), 673–680. (Russian)

The author proves a theorem on infinite regular p-groups which is a generalization of the Theorem of Prüfer for abelian groups. The hypotheses are the following: 1) G is a regular p-group; 2) G is countable; 3) any subgroup of G generated by two elements is finite; 4) the lower layer of G lies in the center; 5) G has no elements of infinite height. The conclusion is that G possesses a uniqeuness basis. *Marshall Hall, Jr.* (Columbus, Ohio).

Scott, W. R. On a result of B. H. Neumann. Math. Z. **66** (1956), 240.

Let X be the set of all groups in which classes of conjugate subgroups are always finite, and let Y be the set of all groups with the property that each subgroup has finite index in its normal closure. B. H. Neumann [Math. Z. **63** (1955), 76–96; MR **17**, 234] shows, among other things, that XCY. The author gives a direct and concise proof of this result. *F. Haimo.*

Suprunenko, D. A. On maximal commutative subalgebras of the full linear algebra. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 3(69), 181–184. (Russian)

The main result of this note is that for $n>6$ the full linear group $\mathrm{GL}(n, P)$ over an algebraically closed field P of characteristic 0 contains infinitely many non-conjugate maximal abelian subgroups. This is an immediate con-

sequence of the fact proved here that for $n>6$ the set of non-isomorphic maximal commutative nilpotent subalgebras of class 3 of the full matrix algebra P_n is infinite. This is in striking contrast to the cases $n=2, 3, 4, 5$ which have been treated extensively by B. Charles [J. Math. Pures Appl. (9) **33** (1954), 81–145; MR **16**, 439]. For each of these four values of n the algebra P_n contains only a finite number of non-conjugate maximal nilpotent commutative subalgebras. The behaviour in the remaining case $n=6$ is still an open problem.

K. A. Hirsch (London).

Fuchs, L. On a useful lemma for abelian groups. Acta Sci. Math. Szeged **17** (1956), 134–138.

Let G be an abelian group with subgroup H and with subgroup M maximal with respect to the property of being disjoint with H. Then the author shows that G is an extension of the direct sum $H+M$ by a torsion group in which the subgroup of elements of order a prime p is isomorphic to the group $(\{pG, M\}\cap H)/pH$. He shows that the power of $G/(H+M)$ is bounded above by the product of the smallest infinite cardinal and the power of H and that, if H is infinite cyclic, then the above quotient is essentially a subgroup of the group of finite rotations of a circle. Various applications of this lemma are given. [See T. Szele, J. London Math. Soc. **28** (1953), 247–250; MR **14**, 721.] *F. Haimo* (St. Louis, Mo.).

de Groot, J. An isomorphism criterion for completely decomposable abelian groups. Math. Ann. **132** (1956), 328–332.

Theorem: Let G, H be abelian groups each of which is a direct sum of groups of rank one, and suppose that each is isomorphic to a pure subgroup of the other. Then G and H are isomorphic. The technique is to sharpen the usual proof of the uniqueness of the decomposition. *I. Kaplansky.*

Schenkman, Eugene. A splitting theorem and the principal ideal theorem for some infinitely generated groups. Proc. Amer. Math. Soc. **7** (1956), 870–873.

Let G be a group, G' its derived group, G^* the intersection of the terms of the lower central series of G. The first theorem of the paper generalizes to certain infinite groups a recent result of the author on finite soluble groups [same Proc. **6** (1955), 286–290; MR **16**, 898]. If G is periodic, G^* abelian and of finite index n in G, and if the Sylow p-subgroups of G^* for $p|n$ have no elements of infinite height, then G^* is complemented in G by a finite nilpotent subgroup H of G, i.e. $G=\{G^*, H\}$ and $G^*\cap H=1$. The second theorem is an extension of the Principal Ideal Theorem. If G is a group in which G' is abelian and of finite index n, and if the Sylow p-subgroups of G' for $p|n$ have no elements of infinite height, then the transfer of G into G' is trivial. A simple example shows that the assumption about elements of infinite height is necessary in both theorems. *K. A. Hirsch* (London).

Ree, Rimhak. The existence of outer automorphisms of some groups. Proc. Amer. Math. Soc. **7** (1956), 962–964.

Schenkman [same Proc. **6** (1955), 6–11; MR **16**, 671] has proved that every finite nilpotent group of class 2 has an outer automorphism. Let N be a group with an element a of order n. If a periodic extension G of N by a cyclic group of order n can be found with the orders all dividing n and with a in the center of G, then the author shows that G possesses an outer automorphism which induces the identity on N and on G/N. Let G be a nilpotent group of class less than some odd prime p. Let G be an extension of a group N with elementary p-group as center by a group of order p. Then G possesses an outer automorphism as above. Finally, he shows that every torsion free nilpotent group with the maximal condition on subgroups has an outer automorphism. *F. Haimo.*

Littlewood, D. E. Plethysm and the inner product of S-functions. J. London Math. Soc. **32** (1957), 18–22.

In a former paper [same J. **31** (1956), 89–93; MR **17**, 583] the author obtained the formula

$$(\{\lambda\}\{\mu\})\circ\{\nu\}=\sum\Gamma_{\alpha\beta\nu}(\{\lambda\}\circ\{\alpha\})(\{\mu\}\circ\{\beta\}),$$

where $\{\alpha\}\{\beta\}=\sum\Gamma_{\alpha\beta\nu}\{\nu\}$. Setting $\{\lambda\}=\{\mu\}$, the further reduction

$$(\{\lambda\}\{\lambda\})\circ\{\nu\}=(\{\lambda\}\otimes\{2\})\circ\{\nu\}+(\{\lambda\}\otimes\{1^2\})\circ\{\nu\}$$

enters the picture and the method of Robinson and Taulbee [Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 723–726; MR **16**, 110] is applied to reduce terms of the form $(\{\theta\}\otimes\{\varphi\})\circ\{\nu\}$. *G. de B. Robinson* (Toronto, Ont.).

Norton, D. A.; and Stein, Sherman K. Cycles in algebraic systems. Proc. Amer. Math. Soc. **7** (1956), 999–1004.

Soit $Q(*)$ un quasigroupe fini, d'ordre n, $a*b=c$ ($a, b, c \in Q$). On forme de proche en proche les produits, ou „points" $b*a=d$, $c*d=a$, $a*f=e$, etc., obtenus en permutant les deux premiers éléments (a, b), puis les deux derniers (a, d), puis le 1er et le 3ème (e, a) et en recommençant. Une telle série se reproduit périodiquement et s'appelle un „cycle". Le nombre des points successifs d'un cycle est multiple de 3, sauf si a et b sont égaux et idempotents, le cycle est alors monome. La „longueur" d'un cycle non monome est le tiers du nombre de ses points.

(3.1) Les cycles d'un loop, contenant l'unité de ce loop, sont en correspondance biunivoque avec les séquences d'inverses à gauche [Artzy, mêmes Proc. **6** (1955), 448–453; MR **16**, 1083]. (4.1) Pour que $Q(*)$, idempotent, vérifie $(a*b)*(b*a)=a$, il faut et il suffit que tous ses cycles, non monomes, aient pour longueur 1. (4.2) L'ordre d'un tel Q est $\equiv 0$ ou 1 (mod 4). (5.1) Pour que tout cycle, non monome, de Q contienne exactement 3 éléments inégaux, il faut et il suffit que Q satisfasse à la loi bilatère des keys [M. Takasaki, Tôhoku Math. J. **49** (1943), 145–207; MR **9**, 8, 735]. (5.2) Un tel Q, idempotent, d'ordre n, existe si $n\equiv 1$ ou 3 (mod 6) et seulement dans ce cas. (6.1) Si tout cycle de $Q(*)$ est de longueur 1 ou 2,

$$(a*f)*(b*a)=a \rightleftarrows (a*b)*(f*a)=a.$$

(6.2) Si Q est commutatif, tous ses cycles ont pour longueur 1 ou 2. (6.3) Structure des cycles lorsque Q obéit à la loi II des keys à droite. (6.4) Contruction d'un tel Q à partir d'un groupe abélien. (6.5) Existence d'un tel Q idempotent pour tout ordre impair. (6.6) Conclusions symétriques pour la loi des keys à gauche.

Plusieurs de ces théorèmes sont les applications aux cycles de propriétés déjà établies par le reviewer [cf. Quasigroupes, Marseille, 1950; MR **13**, 203], à savoir: les no. 10 et 11 pour le th. (5.1); no. 6 et 7 pour (5.2); no. 53 pour (6.3); no. 116, avec $p+1=q=0$ pour (6.4) et $k=2$; no. 52 et 57 pour (6.5).

Page 999, ligne 23, lire $P_1P_3\not t$, θt, au lieu de $P_1P_3\not\theta t$; p. 1001, ligne 15, lire (4.2) au lieu de 3; p. 1004, ligne 23, lire (4.1) au lieu de (3.1); dernière ligne, il faut que le groupe soit d'ordre pair. *A. Sade* (Marseille).

See also: Ghika, p. 638; Terada, p. 644; Sielaff, p. 639; Mycielski, p. 661; Wilson, p. 686; Rogers, p. 686.

THEORY OF NUMBERS

General Theory of Numbers

Venkataraman, L. V. On multigrade equations of degree 3 and order δ. J. Sci. Res. Banaras Hindu Univ. **6** (1955–56), 211–213.

The author observes that it is known that the identity in t

$$(t+x_1)^3+(t-x_1)^3+(t+y_1)^3+(t-y_1)^3=(t+x_2)^3+(t-x_2)^3+(t+y_2)^3+(t-y_2)^3$$

follows from $x_1^2+y_1^2=x_2^2+y_2^2$. Using facts about integers representable as the sum of two squares, he shows how to obtain an infinite number of equations of degree 3 of the form stated above. A number of numerical examples are given. *R. D. James* (Vancouver, B.C.).

Obláth, Richard. Une propriété des puissances parfaites. Mathesis **65** (1956), 356–364.

If the digit a is written k times in juxtaposition, is it possible to find a $k>1$ so that the number $\underbrace{aa\cdots a}_{k}$, interpreted in the denary scale with be a perfect nth power? For the numbers $a=2, 4, 5, 6$ the answer is easily seen to be in the negative. For $a=7$ the proof of the impossibility is more involved and is based on results by T. Nagell and W. Ljunggren. The impossibility for $a=3$ and $a=8$ depends on the fact that $\underbrace{11\cdots1}_{k}$ can never be a cube. For $a=1$ the author proves that $\underbrace{11\cdots1}_{k}$ is never a power if k is even or divisible by 3. It is never a fifth power if k is divisible by 5 and is never a seventh power if k is divisible by 7. Some results on bases other than 10 are given. *I. A. Barnett* (Cincinnati, Ohio).

Lehmer, D. H. A low energy proof of the reciprocity law. Amer. Math. Monthly **64** (1957), 103–106.

The author gives an extremely elegant proof of the quadratic reciprocity law. The proof is very short comprising a little over a printed page. He restates the reciprocity law in the form: the symbols $(p|q)$ and $(q|p)$ are equal except when the odd primes p and q are both of the form $4x-1$. The proof is made by an application of Gauss's lemma and a modification of geometrical considerations used by Eisenstein. *I. A. Barnett.*

Chapron, R. Sur une proposition erronée de Korselt relative aux nombres composés m qui divisent a^m-a. Bull. Sci. Math. (2) **80** (1956), 81–83.

Let $m=p_1^{\alpha_1}p_2^{\alpha_2}\cdots$ be the prime power representation of m. The author interprets a statement of A. Korselt [Intermédiare des Math. **6** (1899), 143] to mean that if $a^m=a \pmod m$ for some a relatively prime to m, then $\alpha_i=1$ and $p_i-1|m-1$ for $i=1, 2, \cdots$. He gives the counter-examples $a=2$, $m=1093^2$ and $a=2$, $m=645$ to show that under his interpretation both parts of Korselt's statement are false. *L. Moser* (Edmonton, Alta.).

Oppenheim, A. On the Diophantine equation

$$x^2+y^2+z^2+2xyz=1.$$

Amer. Math. Monthly **64** (1957), 101–103.

The solution of the equation $x^2+y^2+z^2+2xyz=1$ in rationals was recently given by the reviewer [same Monthly **62** (1955), 251–252]. The present note gives the most general solution in rational integers. If the triple of integers p, q, r has a g.c.d. equal to unity, with one of these integers equal to the sum of the other two, then $x=\pm\cosh p\theta$, $y=\pm\cosh q\theta$, $z=\pm\cosh r\theta$, where

$$\theta=\log(u+(u^2-1)^{\frac{1}{2}}),$$

u an integer ≥ 1, gives the most general non-trivial solution in rational integers. The ambiguous signs must be chosen so that $xyz=-1$. [See also Marsh, ibid. **64** (1957), 122, wherein the most general solution is found by reducing the problem to a Pell equation.] *I. A. Barnett* (Cincinnati, Ohio).

Mordell, L. J. Note on the integer solutions of

$$z^2-k^2=ax^3+by^3.$$

Ganita **5** (1954), 103–104 (1955).

The author considers the equation in the title in the case in which $k=4abc^3$, and proves that in this case there are infinitely many solutions in integers (x, y, z). The proof consists in transforming the given equation into

$$A(t^3+u^3)+Bv^3=Bc^3$$

which by the author's previous paper [J. London Math. Soc. **30** (1955), 111–113; MR **16**, 798] has an infinity non-trivial integer solutions. The reader should be on the look out for two misprints. *D. H. Lehmer.*

McCarthy, Paul J. The representation of one quadratic form by another in valuated fields. Portugal. Math. **14** (1955), 31–34.

Soit K un corps discrètement valué complet. Soit (α, β) où α, β sont ϵK le symbole égal à $+1$ ou à -1 selon que l'équation $\alpha x^2+\beta y^2=1$ a ou n'a pas de solution dans K. Sous les hypothèses: A) la carcatéristique résiduelle p de K est $\neq 2$; B) si E est le groupe des unités de K et si $E^{(2)}$ est celui de leurs carrés, $(E:E^{(2)})=2$; C) si $\alpha, \beta \epsilon E$, on a $(\alpha, \beta)=1$ {remarque du référant; en vertu du lemme de Hensel, cette condition équivaut à la résolubilité, dans le corps résiduel R de K, de $\bar{\alpha}x^2+\bar{\beta}y^2=1$ pour tous $\bar{\alpha}, \bar{\beta} \epsilon R$}, Durfee a montré [Bull. Amer. Math. Soc. **54** (1948), 338–351; MR **9**, 561] que: 1) $(\alpha, \beta)=(\beta, \alpha)$; 2) $(\alpha, -\alpha)=1$; 3) $(\alpha\gamma^2, \beta\gamma^2)=(\alpha, \beta)$; 4) $(\alpha, \beta)(\alpha, \gamma)=(\alpha, \beta\gamma)$; 5) $(\alpha\gamma, \beta\gamma)=(\alpha, \beta)(\gamma, -\alpha\beta)$. L'auteur démontre, sous les mêmes hypothèses, que si $\alpha=\alpha_1\pi^a$, $\beta=\beta_1\pi^b$, où (π) est l'idéal premier de K, on a

$$(\alpha, \beta)(-1|\pi)^{ab}(\alpha_1|\pi)^b(\beta_1|\pi)^a,$$

où $(\mu|\pi)$ est le symbole des restes quadratiques dans K. Il exprime, ensuite, la condition, nécéssaire et suffisante pour qu'une forme quadratique nondégénérée de $n>m$ variables représente une forme quadratique non-dégénérée donnée de m variables. *M. Krasner* (Paris).

Cellitti, Carlo. Sopra una costruzione di sistemi di rappresentanti di classi di forme quadratiche binarie, primitive di prima e di seconda specie rispettivamente, di determinante $D=1 \pmod 8$. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **21** (1956), 57–60.

The author continues his partial rediscovery of one of the standard proofs of the relationship between the class numbers of properly and improperly primitive binary quadratic forms [Boll. Un. Mat. Ital. (3) **10** (1955), 527–530; MR **17**, 827] by showing that they are equal when

the discriminant is congruent to 1 modulo 8. As before, the latest reference to the literature is to a paper of 1840.

J. W. S. Cassels (Cambridge, England).

Oeconomou, Georges. Sur le premier cas du théorème de Fermat pour les exposants pairs. C. R. Acad. Sci. Paris **243** (1956), 1588–1591.

The impossibility of (1) $x^{2n}+y^{2n}=z^{2n}$, $(xyz, n)=1$, n odd, in integers is investigated. The result is that (1) is impossible for a prime q, $(n, q-1)=1$, $(n/q)=-1$ or $n\neq 8v+1$. By the aid of this criterion the author calculated that (1) is impossible for all $n<100000$, with the exception of possibly two numbers. Finally, it is mentioned that (1) is impossible for an infinity of primes n.

N. G. W. H. Beeger (Amsterdam).

See also: Carlitz, p. 642; Lochs, p. 643; Lapin, p. 644; Hughes, p. 666; Shimura, p. 673.

Analytic Theory of Numbers

Pan, Cheng-Tung. On $\sigma(n)$ and $\varphi(n)$. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 637–638.

Let $\sigma(n)$ denote the sum of the divisors of n, and $\phi(n)$ Euler's totient function. Writing

$$\sum_{n\leq x}\sigma(n)=\frac{\pi^2}{12}x^2+\Delta(x),\quad \sum_{n\leq x}\varphi(n)=\frac{3}{\pi^2}x^2+R(X),$$

the author announces the following theorems. Theorem 1.

$$\Delta(x)=O\{x(\log x\log\log x)^{3/4}\}.$$

Theorem 4.

$$\sum_{n\leq x}\frac{R(n)}{n}=\frac{3}{\pi^2}x+O\{x\exp(-A\log^{4/7-\epsilon}x)\},$$

where A is a positive number and ϵ an arbitrarily small positive number.

Theorem 1 improves a result of Walfisz and Davenport [see, e.g., Walfisz, Math. Z. **26** (1927), 66–88]. Theorem 4 improves a result of Pillai and the reviewer [J. London Math. Soc. **5** (1930), 95–101]. *S. Chowla.*

Malaviya, Bimal Kumar. Extension and generalization of Ramanujan's formula on Riemann's zeta function. J. Sci. Res. Banaras Hindu Univ. **6** (1955–56), 206–210.

The formula

$$\frac{\zeta^4(s)}{\zeta(2s)}=\sum_{n=1}^{\infty}\frac{\{d(n)\}^2}{n^s},$$

where $d(n)$ is the number of divisors of n, was proved by Ramanujan. This is now generalized to

$$\frac{\zeta^{k+1}(s)}{\zeta(2s)}=\frac{2}{(k-1)!}\sum_{n=1}^{\infty}\frac{b_1(n)b_2(n)\cdots b_{k-2}(n)b_{\frac{1}{2}k-\frac{1}{2}}(n)}{n^s}$$

where $b_l(n)=(l_1+l)\cdots(l_r+l)$ if $n=p_1^{l_1}\cdots p_r^{l_r}$.

E. C. Titchmarsh (Oxford).

Leech, John. Note on the distribution of prime numbers. J. London Math. Soc. **32** (1957), 56–58.

Let $\pi_j(x)$ be the number of primes $1<p\leq x$ of the form $4n+j$. It is known that $\pi_3(x)-\pi_1(x)$ changes sign infinitely often for sufficiently large x. The author finds that $\pi_3(x)-\pi_1(x)$ is negative for $x=26{,}861$ and for certain values of x between 616,000 and 634,000; the extreme value -8 occurs for $x=623{,}681$. The investigation was carried to $x=3{,}000{,}000$.

Let $\pi_l(x)$ be the number of Gaussian primes $a+bi$ with $a\geq 1$, $b\geq 0$, $1<a^2+b^2\leq x$. Then

$$\pi_l(x)=2\pi_1(x)+\pi_3(\sqrt{x})+1\sim\mathrm{li}(x).$$

Now $\pi(x)-\mathrm{li}(x)$ is predominantly negative, and so is $\pi_l(x)-\mathrm{li}(x)$, but the author has found values of x between 615,000 and 626,000 for which $\pi_l(x)-\mathrm{li}(x)>0$; the extreme value 19.5 occurs for $x=617{,}537$. The prime counts were done on the EDSAC (Cambridge) using tapes of differences between consecutive primes constructed by J. C. P. Miller. *J. L. Selfridge.*

Carlitz, L. Sets of primitive roots. Compositio Math. **13** (1956), 65–70.

Let $f_1(x), \cdots, f_r(x)$ be polynomials (mod p), which are square-free and relatively prime in pairs. Let N_r denote the number of integers x (mod p) for which each of $f_1(x), \cdots, f_r(x)$ is a primitive root (mod p). Then, provided r is fixed and the degrees of the polynomials are fixed, it is shown that

$$p^{r-1}N_r\{\varphi(p-1)\}^{-r}\to 1 \text{ as } p\to\infty$$

through primes. Some variations on this result are also proved. The proofs are straightforward: Satz 496 of Landau's Vorlesungen über Zahlentheorie [Bd. 2, Hirzel, Leipzig, 1927] together with the reviewer's estimates for character sums [Acta Math. **71** (1939), 99–121; MR **1**, 41], or the deeper results of André Weil [Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 345–347; MR **2**, 345] which superseded them, could be used. *H. Davenport* (London).

Wirsing, Eduard. Über die Zahlen, deren Primteiler einer gegebenen Menge angehören. Arch. Math. **7** (1956), 263–272.

Let $\mathfrak{T}$ be a subset of the primes whose relative density in the set of all primes is τ, let $\mathfrak{M}$ be the set of all integers whose prime factors are in $\mathfrak{T}$, and let $M(x)$ denote the number of integers of $\mathfrak{M}$ not exceeding x. From a consideration of special cases [in particular, Landau, Handbuch der Lehre von der Verteilung der Primzahlen, Bd. 2, Teubner, Leipzig-Berlin, 1909, pp. 641–669], it might be expected that $M(x)\sim cx/(\log x)^{1-\tau}$, where c is a constant. The author proves that this is so if and only if $\sum 1/p=\tau\log\log x+c'+o(1)$, where the summation is over all $p\in\mathfrak{T}$, $p\leq x$, and c' is a constant. He also proves, without assuming the result for $\sum 1/p$, that there is an explicit asymptotic formula for $M(x)$. The proof uses Tauberian theorems for infinite series and a theorem on Laplace transforms. *R. D. James* (Vancouver, B.C.).

Meinardus, Günter. Zur additiven Zahlentheorie in mehreren Dimensionen. I. Math. Ann. **132** (1956), 333–346.

The author's general theme is the extension of linear partition problems for rational integers to more general systems. He observes that, in extensions of such problems to an algebraic number field, only the additive group of the field is relevant. He therefore proposes to formulate his basic theorems in terms of vector spaces. This first paper is devoted to a generalization of the Hardy-Ramanujan asymptotic formula for $\log p(m)$, where $p(m)$ is the number of unrestricted partitions of m. The proof is by an extension to an n-dimensional vector space of the Tauberian method of Hardy and Ramanujan [Proc. London Math. Soc. (2) **16** (1917), 112–132]. The general

result is too elaborate to state briefly, but a particular application may be quoted: If $P(\mu)$ is the number of partitions of the (totally positive) integer μ of a totally real number field of degree n into totally positive integers of the field, then

$$\log P(\mu)=(n+1)\left\{\frac{\zeta(n+1)}{\sqrt{d}}N\mu\right\}^{1/(n+1)}(1+o(1))$$

when $N\mu\to\infty$. *A. E. Ingham* (Cambridge, England).

Hodges, John H. Weighted partitions for symmetric matrices in a finite field. Math. Z. **66** (1956), 13–24.

For $\alpha\in GF(q)$, $q=p^f$, p odd, put $c(\alpha)=e^{2\pi i t(\alpha)/p}$, $t(\alpha)=\alpha+\alpha^p+\cdots+\alpha^{p^{f-1}}$. If $A=(\alpha_{ij})$ is a square matrix with elements in $GF(q)$, put $\sigma(A)=\sum_i \alpha_{ii}$. The paper is concerned with the weighted sum

$$S=S(B,U,A)=\sum_{X'AX=B} e\{\sigma(U'X+X'U)\},$$

where A is a non-singular symmetric matrix of order m, B symmetric of order t, U arbitrary $m\times t$, and the summation is over all $m\times t$ matrices X such that $X'AX=B$; all matrices have elements in $GF(q)$. [For the corresponding problem for arbitrary matrices, see Hodges, Duke Math. J. **23** (1956), 545–552; MR **18**, 113.]

Generalized Kloosterman sums $K_n(B,W)$, $L_n(B,W)$ are defined by means of

$$K_n(B,W)=\sum_{|C|\neq 0} e\{-\sigma(BC+WC^{-1})\},\quad K_n(B)=K_n(B,I),$$

$$L_n(B,W)=\sum_{|C|\neq 0}\lambda(C)e(-\sigma(BC+WC^{-1})\},\ L_n(B)=L_n(B,I),$$

the summation extending over non-singular symmetric C of order n; also if $T'AT=\mathrm{diag}(\alpha_1,\cdots,\alpha_r,0,\cdots,0)$, T non-singular, then $\lambda(A)=\psi(\alpha_1\cdots\alpha_r)$, where $\psi(\alpha)=1$ or -1 according as α is a square or non-square of $GF(q)$. When $W=0$, K_n and L_n can be expressed in terms of the sum

$$H(B,r,\lambda)=\sum_C e\{-\sigma(BC)\},$$

where the summation is over all symmetric C of order n, rank r and $\lambda(C)=\lambda$. The main result of the paper is stated in the form

$$S=q^{-\frac{1}{2}t(t+1)}(S_1+S_2+S_3),$$

where

$$S_1=\begin{cases}q^{mt} & (U=0)\\ 0 & (U\neq 0),\end{cases}$$

$$S_3=\begin{cases}\lambda^t(A)\psi^{nt}(-1)q^{nt}K_t(B,W) & (m=2n)\\ \lambda^t(A)\psi^{nt}(-1)q^{nt}G^t(1)L_t(B,W) & (m=2n+1),\end{cases}$$

where $W=U'A^{-1}U$ and $G(1)$ is a Gauss sum; finally S_2 is also exhibited as a sum of generalized Kloosterman sums.

A number of properties of K_n and L_n are obtained; in particular explicit expressions are given for $K_n(B,0)$ and $L_n(B,0)$. In conclusion, the writer shows that the sum

$$H^*(B,U,A,r,\lambda)=\sum_X e\{\sigma(2U'X)\}H(X'AX-B,r,\lambda),$$

the summation extending over all $m\times t$ matrices X, satisfies

$$H^*(B,U,A,r,\lambda)=\sum_D H(D,r,\lambda)S(B+D,U,A),$$

where the summation is now over all symmetric D of order t.

L. Carlitz (Durham, N.C.).

Lochs, Gustav. Über die Unregelmässigkeit der Abstände aufeinander folgender Zahlen mit gegebenen Primfaktoren und einen damit zusammenhängenden allgemeineren Satz. Arch. Math. **7** (1956), 259–262.

The author proves the following result: Let α_i $(i=1,\cdots,k)$ be k given positive numbers with $\alpha_i<\alpha_{i+1}$ and such that no relation $\sum_{i=1}^k a_i\alpha_i=0$ holds with rational a_i. Consider all numbers of the form $\sum_{i=1}^k x_i\alpha_i$, where the x_i are non-negative integers, and arrange them in an increasing sequence. Denote the nth term of the sequence by λ_n and let $\lambda_{n+1}-\lambda_n=\mu_n$, $\mu_n\mu_{n+1}=c_n$. Then

(1) $\limsup c_n\geq 1$ and $c_n>1$ infinitely often,

(2) $\liminf c_n\leq 1$,

(3) $\limsup c_n-\liminf c_n\geq(3-\sqrt{5})/2$.

In particular, if $\alpha_i=\log p_i$ $(i=1,\cdots,k)$, where the p_i are distinct primes, then $\lambda_n=\log z_n$, where z_n is the nth term of the sequence of integers whose only prime factors are the p_i. In this case, $(z_{n+1}-z_n)/(z_{n+2}-z_{n+1})\sim c_n$ and (3) shows that, although the difference $z_{n+1}-z_n\to\infty$ [W. Specht, S-B. Math. -Nat.Kl. Bayer. Akad. Wiss. **1948**, 149–169; MR **11**, 500], it behaves in an irregular manner. For $k=2$ better results than (1)–(3) are proved and the author suggests that they may also be true for $k\geq 3$.

R. D. James (Vancouver, B.C.).

See also: Smith, p. 635; Djerasimović, p. 635; Rivkind, p. 680.

Theory of Algebraic Numbers

★ **Masuda, Katsuhiko. On the arithmetic on a Galois structure.** Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 243–245. Science Council of Japan, Tokyo, 1956.

Discusses a generalization of the Galois structures, introduced by Hasse [J. Reine Angew. Math. **187**, (1949), 14–43; MR **11**, 576] and studied by Masuda [ibid. **193** (1954), 161–165; MR **16**, 790], in which the group and field involved are topological. *G. Whaples.*

★ **Iwasawa, Kenkichi. Galois groups acting on the multiplicative groups of local fields.** Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 63–64. Science Council of Japan, Tokyo, 1956.

A summary of results already published [Trans. Amer. Math. Soc. **80** (1955), 448–469; MR **17**, 715].

G. Whaples (Bloomington, Ind.).

Kubota, Tomio. Über den bizyklischen biquadratischen Zahlkörper. Nagoya Math. J. **10** (1956), 65–85.

Let k be an algebraic extension of the rational field whose Galois group is the four-group. S. Kuroda [J. Fac. Sci. Imp. Univ. Tokyo Sect. I. **4** (1943), 383–406; MR **9**, 12] showed that if K is real, and ε_i are generating units for the quadratic subfields k_i of K, then the unit group of K has a set of generators of one of seven types: $\{\varepsilon_1,\varepsilon_2,\varepsilon_3\}$, $\{(\varepsilon_1)^t,\varepsilon_2,\varepsilon_3\}$, $\{(\varepsilon_1\varepsilon_2)^t,\varepsilon_2,\varepsilon_3\}$ etc. Kubota proves that each of these types occurs for infintely many fields K, and proves a corresponding result for complex K. He then studies the ideal class groups of K and the k_i, proving by class field theory methods a relation, due to Dirichlet, Herglotz and Kubota [Kubota, Nagoya Math. J. **6** (1953), 119–127; MR **15**, 605] which expresses class num-

ber of K in terms of class numbers of the k_i and the index in unit group of K of the subgroup generated by the units of the subfields k_i. *G. Whaples.*

★ **Tannaka, Tadao. On the generalized principal ideal theorem.** Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 65–70. Science Council of Japan, Tokyo, 1956.

Let k be an algebraic number field and K/k normal with Galois group G. Call an ideal $\mathfrak{A}$ in K ambiguous if there exist elements $A_\sigma \in K$ with $\mathfrak{A}^{1-\sigma}=(A_\sigma)$ (principal ideal) and $A_\tau A_\sigma^\tau / A_{\sigma\tau}=\varepsilon_{\sigma,\tau}=\varepsilon_{\sigma,\tau}$ for all σ, τ in G. Tannaka proves that if K is the absolute class field for k then every ambiguous K-ideal is principal. From this it follows that if E is the unit group of K, and $H'^2(G, E)$ is the group of all 2-cohomology classes containing $\{\varepsilon_{\sigma,\tau}\}$ with $\varepsilon_{\sigma,\tau}=\varepsilon_{\tau,\sigma}$, then $H'^2(G, E)=1$ when K/k is absolute class field. The proof depends on the generalized principal ideal theorem: for K/k as above and Ω/k a cyclic intermediate field then every ambiguous (in ordinary sense) ideal in Ω becomes principal in K. [Tannaka and Terada, Proc. Japan Acad. **25** (1949), no. 8, 7–8; Tannaka, ibid. **25** (1949), no. 11, 26–31; MR **12**, 316]. The paper also includes a proof of this theorem. Similar results are proved also for the case when K/k is a "Strahl" class field. *G. Whaples.*

Terada, Fumiyuki. A generalization of the principal ideal theorem. J. Math. Soc. Japan **7** (1955), 530–536.

Let G be a finite group, S an automorphism of it, and H the invariant subgroup generated by the commutator subgroup of G together with all elements $S(\sigma)\sigma^{-1}$, with $\sigma \in G$. Using the methods of Chevalley [Class field theory, Nagoya Univ., 1954; MR **16**, 678], Terada defines mappings, depending on S, of various cohomology groups of G and H, using them to prove Theorem 1: If G is the Galois group of $\bar{\Omega}/\Omega$, if K is the subfield corresponding to H, if Ω is an algebraic number field, and if $C_{\bar{\Omega}}$, C_Ω denote idèle class groups, then the kernel of a certain endomorphism $j_0 S_0^*$ of $C_\Omega/N_{\bar{\Omega}/\Omega}C_{\bar{\Omega}}$ is contained in $N_{\bar{\Omega}/K}C_{\bar{\Omega}}$. If k is an algebraic number field, K/k and $\bar{\Omega}/K$ absolute class fields, and Ω/K cyclic with generating automorphism s, this specializes to Theorem 2: Every ambiguous idèle class in C_Ω (i.e. every c with $c^{1-s}=1$) is contained in $N_{\bar{\Omega}/K}C_{\bar{\Omega}}$. This is the generalized principal ideal theorem of Tannaka and Terada. [See the preceding review.] *G. Whaples* (Bloomington, Ind.).

Lapin, A. I. Letter to the Editor. Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 583–584. (Russian)

L'auteur donne la rectification des démonstrations des théorèmes 1 et 8 dans Izv. Akad. Nauk SSSR. Ser. Mat. **17** (1953), 31–50 [MR **15**, 938] où le lemme 1 de ce travail a été appliqué dans les cas, où ses hypothèses n'ont pas été vérifiées, ainsi que celle des démonstrations des théorèmes analogues [ibid. **18** (1954), 145–158; MR **15**, 938], où le même lemme est employé. Ces détails de démonstration sont trop techniques pour qu'on puisse en parler ici. *M. Krasner* (Paris).

Iwasawa, Kenkichi. A note on class numbers of algebraic number fields. Abh. Math. Sem. Univ. Hamburg **20** (1956), 257–258.

Let k be an algebraic number field, K/k finite normal; let h and H be class numbers of k and K. Iwasawa proves: (I) If there exists a prime divisor P of k which is fully ramified in K/k, then $h|H$. (II) If furthermore K/k is cyclic of p-power degree and has no ramified prime divisor except P, then $p|H \Rightarrow p|h$. (III) If p is a regular prime the class number of an abelian field with conductor p^e is prime to p. He mentions that (I) has been proved by C. Chevalley [C. R. Acad. Sci. Paris **192** (1931), 257–258].

G. Whaples (Bloomington, Ind.).

See also: McCarthy, p. 641; Lang, p. 672; Shimura, p. 673.

ANALYSIS

Functions of Real Variables

Viterbo, Guido. Su una estensione del concetto di derivata. Ricerca, Napoli **6** (1955), 11–21.

Let f be defined in a set X with c (finite or infinite) as a point of accumulation. If $\lim_{x\to c} f(x)$ exists and is finite, the author calls it the derivative of order zero of f at c and designates it by $f^0(c)$. Let ε denote $x-c$ if c is finite and $1/(a-x)$ (a being a finite constant) if c is infinite. Then if $\lim_{x\to c}(f(x)-f^0(c))/\varepsilon$ exists and is finite, it is called the first derivative $f'(c)$ of f at c. Similarly $f''(c)$ is defined by the finite

$$\lim_{x\to c}\frac{1}{\varepsilon}\left(\frac{f(x)-f^0(c)}{\varepsilon}-f'(c)\right)$$

and, by induction, $f^{(n)}(c)$ is defined by the finite

$$\lim_{x\to c}\frac{1}{\varepsilon}\left[\frac{1}{\varepsilon}\left\{\cdots\left(\frac{1}{\varepsilon}\left(\frac{f(x)-f^0(c)}{\varepsilon}-f'(c)\right)-f''(c)\right)\cdots\right\}-f^{(n-1)}(c)\right].$$

The derivatives of higher order of f at c can exist if the lower derivatives do exist only at c and not necessarily in a neighborhood of c. The author deisgnates the ordinary nth derivative of f at c by $\overline{f^{(n)}(c)}$. Then $f'(c)$ is more general than $\overline{f'(c)}$. If $\overline{f^{(n)}(c)}$ exists, one has $\overline{f^{(n)}(c)}=n!f^{(n)}(c)$.

A. Rosenthal (Lafayette, Ind.).

See also: Cinquini-Cibrario, p. 655; Oleĭnik, p. 656.

Measure, Integration

Peyovitch, T. Sur quelques théorèmes élémentaires des intégrales généralisées. Comment. Math. Univ. St. Paul. **5**(1956), 77–80.

The principal theorem is that if $f(x)$ is integrable for $x\geq a$ and such that $\int_x^\infty f(t)dt=o(1)$ as $x\to\infty$, then for k any positive constant, $(\phi(x))^k\int_x^\infty f(t)(\phi(t))^{-k}dt=o(1)$. Also $\int_x^\infty (\phi(x))^{k-1}\phi'(x)dx \int_x^\infty f(t)(\phi(t))^{-k}dt=o(1)$ as $x\to\infty$ if $\phi(x)>\infty$ as $x\to\infty$, and has a positive integrable derivative for $x\geq a$. Proof is by use of integration by parts. Other theorems are variants or consequences of this one.

T. H. Hildebrandt (Ann Arbor, Mich.).

Lakshmana Rao, S. K. A generalisation of Dirichlet's multiple integral. Edinburgh Math. Notes **40** (1956), 16–18.

I. J. Good's generalization of the integral of the title [same Notes **38** (1952), 7–8; MR **14**, 959] is established by the use of Mellin transforms.

R. P. Boas, Jr.

Berg, Lothar. Allgemeine Kriterien zur Massbestimmung linearer Punktmengen. Math. Nachr. **14** (1955), 263–285 (1956).

As is well-known there exists an extensive metric theory of diophantic approximations [see J. F. Koksma, Diophantische Approximationen, Springer, Berlin, 1936, pp. 43–49, 116–118]. This theory gives information about the measure of sets of real numbers represented by certain classes of continued fractions or by classes of "decimal" representations.

L. Holzer [Akad. Wiss. Wien. Math.-Nat. Kl. S.-B. IIa. **137** (1928), 421–453] extended this theory to the representation of real numbers by Lüroth series. One of his results is the analogue of Bernstein's theorem, that any positive integer k appears an infinity of times in the sequence of the quotients in the continued fraction algorithm of any real number, if we exclude a set of real numbers of measure zero.

In this thesis Berg generalizes Holzer's theory to a class of expansions which include the series of Lüroth, Engel and Sylvester [for these series cf. O. Perron, Irrationalzahlen, 2. Aufl., de Gruyter, Berlin, 1939, pp. 116–127], but not the continued fraction algorithm. His results are in general too complicated to state here, but he shows for example that the analogue of Bernstein's theorem does not hold for the Engel and Sylvester series.

The Engel series are treated in more detail.

J. Popken (Amsterdam).

Daboni, Luciano. Estensione del teorema di Riemann-Dini, sulle serie, alle funzioni non sommabili nella teoria astratta della integrazione. Riv. Mat. Univ. Parma **6** (1955), 293–300.

The theorem on recording conditionally convergent series is generalized by the author to Lebesgue-Stieltjes integrals in an abstract space. Let f be a μ-measurable function on a μ-measurable set L. Let $L=\mathscr{L}^+ + \mathscr{L}^-$ where $f(x)\geqq 0$ on $\mathscr{L}^+$ and $f(x)<0$ on $\mathscr{L}^-$. Moreover, let $L=L_1+L_2$ be the Hahn decomposition of L with respect to μ (i.e., $\mu(L')\geqq 0$ for every $L'\subset L_1$ and $\mu(L'')\leqq 0$ for every $L''\subset L_2$). Set $L=L^+ + L^-$ where

$$L^+=L_1\mathscr{L}^+ + L_2\mathscr{L}^- \text{ and } L^-=L_1\mathscr{L}^- + L_2\mathscr{L}^+.$$

A set K is called by the author to be of type $(\mathscr{K})$ if $K\subset L$ and f is bounded on K. Set $K^+=KL^+$ and $K^-=KL^-$. Assume f to be non-μ-summable on L^+ and L^-, but to satisfy the following condition on L: For a given $\varepsilon>0$, it is possible to determine a set K_ε of type $(\mathscr{K})$ such that for every set K of type $(\mathscr{K})$ with $K\supset K_\varepsilon$ the set $K-K_\varepsilon$ can be decomposed into finitely many sets H of type $(\mathscr{K})$ for which $\int_H{}^+ f(x)d\mu - \int_H{}^- f(x)d\mu$ be smaller than ε. Then for every given real number a one can construct a non-decreasing sequence of sets $\{K_n\}$ of type $(\mathscr{K})$ whose union is L, such that $\lim_{n\to\infty}\int_{K_n} f(x)d\mu = a$.

A. Rosenthal (Lafayette, Ind.).

Bauer, Heinz. Über die Beziehungen einer abstrakten Theorie des Riemann-Integrals zur Theorie Radonscher Masse. Math. Z. **65** (1956), 448–482.

The author considers an abstract Riemann integral recently treated by the reviewer [Amer. J. Math. **76** (1954), 168–182; MR **15**, 631] from the point of view of the theory of Radon measure. Let L be a vector lattice of real-valued functions on a set S and suppose that $f\cap 1 \in L$ for every $f\in L$. The uniform closure of the set of bounded functions in L is then an algebra A, and, supposing that A does not contain the constant functions, S can be densely imbedded in a locally compact space $\bar{S}$ in such a way that A becomes the algebra of all continuous real-valued functions vanishing at infinity on $\bar{S}$. [$\bar{S}$ can be viewed as the space of regular maximal ideals of the Banach algebra A.] Now let I be any non-negative linear functional defined on L. The functions of L having compact support (considered always on $\bar{S}$) are sufficiently numerous so that I defines a unique Radon measure μ, and $I(f)=\int f\,d\mu$ when every $f\in L$ has compact support. [The unbounded functions of L can also be extended to $\bar{S}$, and become continuous functions except for assuming the values $\pm\infty$ on compact sets of measure 0.] It follows now that all the (extended) functions of L are μ-summable, with $\int f\,d\mu \leqq I(f)$ if $f\geqq 0$. Against this background the author considers various questions of Riemann integrability and the theory of content treated by the reviewer in the above-mentioned paper. In particular, he recovers conditions that will ensure the identity of the functional and the integral over L. Also, in order to throw more light on the possible discrepancies between $I(f)$ and $\int f\,d\mu$, the author investigates the decompositions of I and μ into continuous and singular parts, both on S and on $\bar{S}$.

L. H. Loomis (Cambridge, Mass.).

Cecconi, Jaurès. Una osservazione sulla convergenza in variazione e in area. Boll. Un. Mat. Ital. (3) **11** (1956), 524–525.

A sequence (T_n, A) $(n=1, 2, \cdots)$ of continuous mappings from the closed unit square A into E_3 is said to "converge" toward a mapping (T, A), say $T_n\to T$, if it converges uniformly. It is said to "converge in area" if $T_n\to T$ and $L(T_n)\to L(T)$, where L denotes Lebesgue area. It is said to "converge in area relatively to a plane π in E_3" if $T_n\to T$ and $L(\tau_\pi T_n)\to L(\tau_\pi T)$, where τ_π denotes the orthogonal projection of E_3 onto π. It is known that convergence in area implies convergence in area relatively to every plane π in E_3 (T. Radó, L. Cesari). It is also known that the convergence in area relatively to three orthogonal planes (convergence in variation) does not imply convergence in area. The author proves that convergence in area relatively to every plane π of E_3 implies convergence in area. This result extends to surfaces known results for curves. The proof is based on statements of L. Cesari, T. Radó and R. G. Helsel.

L. Cesari (Lafayette, Ind.).

Neugebauer, Christoph J. A strong cyclic additivity theorem of a surface integral. Riv. Mat. Univ. Parma **6** (1955), 239–259.

A surface integral $J(T)$ has been defined by Cesari [Surface area, Princeton, 1956, App. B; MR **17**, 596] by means of topological and measure-theoretical considerations, for any continuous mapping T: $A\to E_3$, from an admissible set $A\subset E_2$ into E_3 (or E_n), of finite area. If $A=Q$ is a 2-cell, and $T=hg$, g: $Q\Rightarrow M$, h: $M\to E_3$, any decomposition of T into two continuous mappings g, h, denote by $[c]$ the countable collection of all proper cyclic elements c of the Peano space M (middle space) and, for any $c\in[c]$, denote by r_c the monotone retraction of M onto c. The main result of the paper is the following (strong) cyclic additivity theorem: (*) $J(T)=\sum_c J(hr_cg)$ for every $T:Q\to E_3$ of finite area. This theorem extends to the integral J an analogous statement proved by E. J. Mickle and T. Radó [Trans. Amer. Math. Soc. **66** (1949), 347–365; MR **11**, 167] for the area. Also, it extends the cyclic additivity theorem proved for $J(T)$ by J. Cecconi [Rev. Mat. Univ. Parma **4** (1953), 43–67; MR **15**, 111] when $T=hg$ is the monotone-light decomposition of T.

The proof is based on a discussion of the proper cyclic elements of a Peano space, on the quoted result of E. J. Mickle and T. Radó, and on results of Cesari. [For further references on the integral $J(T)$ see V. E. Bononcini, MR **14**, 482; **15**, 945; J. Cecconi, MR **15**, 111, 112; L. Cesari, **9**, 505; **10**, 259; **12**, 86; **14**, 292; App. B of the book quoted above; J. M. Danskin, MR **14**, 292; A. G. Sigalov, MR **13**, 257, 1139.] *L. Cesari* (Lafayette, Ind.).

Bharucha-Reid, A. T. On random elements in Orlicz spaces. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 655–657.

Let (Ω, a, p) be a probability space and $\mathfrak{X}=L_\Phi(\Omega, p)$ an Orlicz space as a space of generalised random variables. The author states the following two theorems for random elements in Orlicz spaces: I. Let $\{x_t(\omega)\}$ be an infinite sequence of independent and identically distributed random elements in L_Φ and let $z_n=1/n\sum_{t=1}^{n} x_t$. If (i) the expectation $\mathscr{E}x_t$ exists, and $\mathscr{E}\|x_t\|=k<\infty$ and the adjoint space of L_Φ is separable then $\lim_{n\to\infty} z_n=\mathscr{E}x_t$. II. Let $\{x_t(\omega)\}$ be a strictly stationary sequence of independent and identically distributed random elements in L_Φ. If (i) $\mathscr{E}x_t$ exists and $\mathscr{E}\|x_t\|$ is finite (ii) the probability measure p is separable and there exists a constant M such that $\Phi(2u)/\Phi(u)\leq M$ if $u\geq 0$ and $\Psi(2v)/\Psi(v)\leq M$ if $v\geq 0$ (where Ψ is the complementary Young function of Φ), then $\lim 1/n\sum_{t=1}^{n} x_t$ exists weakly almost surely and is an element of L_Φ. The proofs will be given in a subsequent paper. *W. A. J. Luxemburg* (Toronto, Ont.).

★ **Fréchet, Maurice. Abstrakte Zufallselemente.** Bericht über die Tagung Wahrscheinlichkeitsrechnung und mathematische Statistik in Berlin, Oktober, 1954, pp. 23–28. Deutscher Verlag der Wissenschaften, Berlin, 1956.

A brief statement of generalisations to metric spaces of theorems now classical in the mathematical theory of probability, such as the law of large numbers. [References to previous work are given, e.g., Doss, Bull. Sci. Math. (2) **73** (1949), 48–72; MR **11**, 190; Fréchet, Rev. Sci. **82** (1944), 483–512; Ann. Inst. H. Poincaré **10** (1948), 215–310; MR **8**, 141; **10**, 311; Mourier, C. R. Acad. Sci. Paris **236** (1953), 575–576; MR **14**, 662]. *I. J. Good.*

See also: van Rootselaar, p. 632; Zmorovič, p. 648; Tsuji, p. 650; Kumar, p. 652; Novák, p. 679; Kloss, p. 680; Rivkind, p. 680.

Functions of Complex Variables

Mori, Akira. On quasi-conformality and pseudo-analyticity. Trans. Amer. Math. Soc. **84** (1957), 56–77.

The paper by Mori was written in 1955 and duplicates some results that had been obtained shortly before that time by the reviewer [J. Analyse Math. **3** (1954), 1–58, 207–208; MR **16**, 348] and by K. Strebel [Proc. Amer. Math. Soc. **6** (1955), 903–909; MR **17**, 473]. Nevertheless, its fresh approach and several interesting side remarks add impetus to the growth of a relatively undeveloped field.

The interest centers around the "almost everywhere" properties of quasiconformal mappings. It is noted that a quasiconformal mapping has a total differential a.e., and it is proved that the mapping is absolutely continuous on almost every horizontal line (also proved by Strebel). The 2-dimensional absolute continuity is stated as an open problem. Several footnotes referring to the existing literature have been added by L. Bers after the author's untimely death. *L. V. Ahlfors* (Cambridge, Mass.).

Bers, Lipman. On a theorem of Mori and the definition of quasiconformality. Trans. Amer. Math. Soc. **84** (1957), 78–84.

In this companion paper to the one reviewed above Bers makes a systematic comparison of the geometric definition of a quasiconformal mapping with the analytic definition, based on L_2-derivatives, and proves that the definitions are equivalent. He points out that the 2-dimensional absolute continuity is an easy consequence of this equivalence. *L. V. Ahlfors* (Cambridge, Mass.).

Nedelcu, Mariana. Le polynôme aréolaire d'ordre n dans l'espace à trois dimensions. Acad. R. P. Romîne. Stud. Cerc. Mat. **7** (1956), 197–237. (Romanian. Russian and French summaries)

Let ψ be a 4×1-matrix, the elements of which are functions of the real variables x, y, z. If the elements belong to class C^1, the areolar derivative of ψ is defined by the symbolic matrix equation

$$D\psi=\begin{bmatrix} 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & 0 & \frac{-\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & 0 & \frac{-\partial}{\partial x} \\ \frac{\partial}{\partial z} & \frac{-\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix}\psi.$$

If the elements do not belong to C^1, $D\psi$ is defined by a more general formula, involving integration over a closed surface. If $D^{n+1}\psi=0$, ψ is called an areolar polynomial of order n. The author studies these polynomials, and obtains a number of theorems which somewhat resemble elementary theorems concerning analytic functions of a complex variable. These results include series representations of the polynomials and representations in terms of integrals taken over the boundary of the domain considered. The author states, but does not elaborate on the remark, that the results are useful in connection with boundary value problems relating to partial differential equations. *L. A. MacColl* (New York, N.Y.).

Vijayaraghavan, T.; and Padmavally, K. On a question of J. M. Whittaker. Proc. Nat. Inst. Sci. India. Part A. **22** (1956), 1–6.

Whittaker [Duke Math. J. **21** (1954), 571–573; MR **16**, 232] asked whether $\sum_1^\infty a_n/(nz)$ can vanish identically with a_n not all zero if f is a transcendental entire function, and showed that it is equivalent to ask whether an entire function of e^z can have an infinity of zero coefficients in its Taylor series. The authors construct an example answering the second question affirmatively. *R. P. Boas, Jr.*

Moppert, K. F. On a property of complex power series. Amer. Math. Monthly **64** (1957), 88–89.

Consider the Hilbert space of complex functions $f(z)$ in the unit circle, K, with inner product

$$(f, g)=\int\int_K f(z)\overline{g(z)}dxdy.$$

If f is approximated in mean by $\sum_{\nu=0}^{n} a_\nu z^\nu$, the Euler-

Fourier formula for a_ν agrees with Cauchy's formula whenever $f(z)$ is regular in K. *R. M. Redheffer.*

Ono, Isao. The total number of zeros and poles in the system of a complex variable. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. **5** (1956), 158–162.

For a vector function

$$W(z)=\begin{pmatrix} w_1(z) \\ \cdot \\ \cdot \\ \cdot \\ w_2(z) \end{pmatrix}$$

the absolute value is defined as $|W(z)|=\{W^*(z)W(z)\}^{\frac{1}{2}}$, where $W^*(z)$ denotes the conjugate transpose of $W(z)$; a point a is a zero if $|W(a)|=0$ and a pole if $\lim|W(z)|=\infty$ as $z\to a$. $W(z)$ is said to be meromorphic in a region D and continuous on the boundary (D) of D if the same is true of each $w_j(z)$ $(j=1, 2, \cdots, k)$. If $n(0)$ and $n(\infty)$ denote respectively the number of zeros and poles of $W(z)$ in D counted with multiplicities, then the Cauchy-type relation holds:

$$n(0)-n(\infty)\leqq\frac{1}{2\pi i}\int_{(D)}|W(z)|^{-2}\{W^*\partial W/\partial z\}dz,$$

the equal sign being valid if and only if $W(z)=f(z)C$, C=constant vector. Likewise if D: $|z|<R(\leqq\infty)$ the Jensen-type relation holds:

$$\int_0^R\{N(0,r)-N(\infty,r)\}dr\leqq\frac{1}{2\pi}\int_0^{2\pi}\log|W(Re^{i\theta})|d\theta-m\log|C|,$$

where $N(x,r)=(1/r)\{n(x,r)-n(x,0)\}$; $n(0,r)$ and $n(\infty,r)$ are the number of zeros and poles of $W(z)$ in $|z|<r$; and $C=\lim z^{-m}W(z)\neq 0$ as $z\to 0$. A generalized form of Green's Theorem is used in the proof. *M. Marden.*

Watson, G. N. A bilinear transformation. Edinburgh Math. Notes no. **40** (1956), 1–7.

The author gives a solution, together with comments, of a problem solved by Cayley [Math. Ann. **15** (1879), 238–240=Collected Math. Papers, v. 10, Cambridge, 1896, pp. 153–154] and Forsyth [Theory of functions of a complex variable, 2nd ed., Cambridge, 1900, p. 717] concerning a certain mapping of the complex plane.

D. R. Hughes (Columbus, Ohio).

Andreian Cazacu, Cabiria. Über die normal ausschöpfbaren Riemannschen Flächen. Math. Nachr. **15** (1956), 77–86.

Let R be a Riemann surface in the sense of Stoïlow which is defined as a covering surface of the Riemann sphere. A domain D contained in the projection of R is called completely ramified if the number of sheets of every island above D is $\geqq 2$. The author proves: Let R be normally exhaustible in the sense of Stoïlow [Compositio Math. **7** (1940), 428–435; MR **2**, 85]. Denote by h the maximum number of disjoint, simply-connected, completely ramified domains, by n the number of sheets of R, by g the genus of R and by c the order of connectivity of R. Then, if $g=0$, $h\leqq 3$; if g and n are finite, $h\leqq 4+4(g-1)/n$; if c and n are finite, $h\leqq 2+2(c-2)/n$; if n is infinite, $h\leqq 4$ or $h\leqq 2$ according as g or c is finite. Furthermore, the author studies interrelations among normal exhaustible Riemann surfaces, Riemann surfaces of the class A_∞ introduced by Volkovyskiĭ [Trudy Mat. Inst. Steklov. **34** (1950); MR **14**, 156] and Riemann surfaces of the class (I) introduced by Stoïlow [Ann. Soc. Polon. Math. **25** (1952), 69–74; MR **14**, 1076]. *K. Noshiro* (Nagoya).

Savage, Nevin. Weak boundary components of an open Riemann surface. Duke Math. J. **24** (1957), 79–95.

A capacity can be assigned to each Kerékjártó-Stoïlow boundary component of an open Riemann surface. The author calls a component "weak" if its capacity vanishes. In the planar case it can be shown that a component is weak if and only if it corresponds to a point under every conformal mapping into the plane. In the general case the author gives sufficient conditions modelled on corresponding known criteria for parabolic surfaces.

L. Ahlfors (Cambridge, Mass.).

Kennedy, P. B. A class of integral functions bounded on certain curves. Proc. London Math. Soc. (3) **6** (1956), 518–547.

The author's main object is to construct examples showing the sharpness of several theorems connected with the Denjoy-Carleman-Ahlfors theorem on asymptotic values of entire functions. He first constructs subharmonic functions with the desired properties and then is able to infer the existence of entire functions with sufficiently nearly the same rate of growth. The principal results are as follows. (I) A subharmonic analogue of the Hadamard factorization theorem, extending to arbitrary finite order ρ the theorem given by Heins [Ann. of Math. (2) **49** (1948), 200–213; MR **9**, 341] for $\rho<1$. (II) Let C_1 and C_2 be receding paths, i.e. Jordan curves joining 0 to ∞ and not intersecting except at 0, satisfying some additional restrictions; let D be one of the domains which they bound, let $w=\phi(z)$ map D conformally on $\Re(w)>0$ with 0, ∞ corresponding to 0, ∞, and let $U=\Re(\phi)$. Let $a>0$. Let M denote maximum modulus and let σ denote maximum value. Then there exists an entire function $f(z)$ with an infinity of zeros, all lying on C_1 and C_2, satisfying $|f(z)|\leqq a$ in the complement of D;

$$\log M(r,f;D)=\sigma(r,U;D)+O(\log r);$$

$$\log\log M(r,f,D)=\log\sigma(r,U;D)+O(r^{-1}\log r);$$

and $\log|f(z)|=U(z)+O(\log|z|)$ uniformly if z stays far enough away from C_1 and C_2. To describe the applications, suppose that there are n receding paths which are asymptotic paths (or indeed something more general) for an entire function f. The Denjoy-Carleman-Ahlfors theorem states that $\beta=\liminf r^{-\frac{1}{2}n}\log M(r,f)>0$; but there are results proved by Ahlfors [Acta Soc. Sci. Fenn. Nova Ser. A. **1**, no. 9 (1930)] and Macintyre [J. London Math. Soc. **10** (1935), 34–39] which state more if the asymptotic paths differ widely from straight lines. Then (III) the author shows that both of these results are best possible. Now suppose that $\beta<\infty$; then Heins [op. cit.] showed that f is of order $\frac{1}{2}n$, conjectured that $\log M(r,f)\sim\beta r^{\frac{1}{2}n}$, and proved this for $n=1$. The author [Proc. London Math. Soc. (3) **5** (1955), 22–47; MR **16**, 809] had obtained weaker conclusions when $\beta<\infty$. He now shows (IV) that two of these are best possible, thus incidentally disproving Heins's conjecture for $n>1$. *R. P. Boas, Jr.*

Mikusiński, J. Sur la croissance de la fonction opérationnelle $\exp(-s^\alpha\lambda)$. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 423–425.

The author investigates the rate of growth of the function $F_\alpha(\lambda,t)=(2\pi i)^{-1}\int_{-\infty}^{\infty}\exp(zt-z^\alpha\lambda)dz$, where $\lambda>0$, $t\geqq 0$, $0<\alpha<1$, which arises in the representation of the

operator $\exp(-S^{\alpha}\lambda)$, where S is the differentiation operator. It is shown that

$$F_{\alpha}(\lambda, t)<\left(\frac{\alpha\lambda}{t}\right)^{1/(1-\alpha)}\exp\left[-(1-\alpha)\alpha^{\alpha/(1-\alpha)}\left(\frac{\lambda}{t^{\alpha}}\right)^{1/(1-\alpha)}\right]$$

$$\text{for}\frac{\lambda}{t^{\alpha}}>\frac{2}{\alpha^{5}}.$$

R. E. Fullerton (College Park, Md.).

Rahman, Qazi Ibadur. On entire functions of infinite order. Tôhoku Math. J. (2) **8** (1956), 165–169.

The author gives an alternative proof of Clunie's theorem [Quart. J. Math. Oxford Ser. (2) **6** (1955), 88–90; MR **16**, 1094] connecting the central index with the maximum modulus of the derivatives of an entire function of infinite order. He also shows that for such functions

$$\liminf\frac{\gamma^{m}}{\nu(r)}\int_{a}^{r}x^{-m-1}\nu(x)dx=0\ \ (m>0),$$

which is stronger than Shah's $\liminf(\nu(r))^{-1}\log\mu(r)=0$ [Math. Student **10** (1942), 80–82; MR **4**, 137].

R. P. Boas, Jr. (Evanston, Ill.).

Rahman, Qazi Ibadur. Maximum modulus and the zeros of an entire function. Ganita **5** (1954), 143–148 (1955).

The author gives a general theorem about limits of indetermination of some expressions involving integrals and uses it to deduce several connections among $\mathrm{Lim}\{\log M(r)\}/\{r^{\rho}L(r)\}$, $\mathrm{Lim}\,n(r)/\{r^{\rho}L(r)\}$, where Lim stands for lim sup or lim inf, $M(r)$ and $n(r)$ refer to an entire function of positive order ρ, and $L(r)$ is a slowly increasing function. *R. P. Boas, Jr.* (Evanston, Ill.).

Rahman, Q. I. On a class of integral functions of zero order. J. London Math. Soc. **32** (1957), 109–110.

If $f(z)$ is an entire function such that

$$\log M(r, f)=O(\log r)^{2-c}\ (0\leqq c<1),$$

then for all x

$$\log M(r, f)\sim N(r, f-x)\sim\log M(r(\log r)^{c}, f)\sim N(r(\log r)^{c}, f-x).$$

R. P. Boas, Jr. (Evanston, Ill.).

Nehari, Z.; and Netanyahu, E. On the coefficients of meromorphic schlicht functions. Proc. Amer. Math. Soc. **8** (1957), 15–23.

Let $f=z^{-1}+a_1z+a_2z^2+\cdots$ map $|z|<1$ conformally on the exterior of a curve which is starlike with respect to the origin. It is shown that $|a_n|\leqq 2/(n+1)$ for $n=3, 4, 5, 6$, and the proof presumably works for larger n. The method depends, as might be expected, on a reduction to inequalities for coefficients of functions with positive real part, but the calculations are quite intricate. Typical of the reasoning is the lemma that when $1+b_1z+b_2z^2+\cdots$ and $1+c_1z+c_2z^2+\cdots$ have positive real part, then so also does $1+(b_1c_1z+b_2c_2z^2+\cdots)/2$. *P. R. Garabedian.*

Mikhail, M. N. On the a-values of the random meromorphic function. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 170–180.

Im Anschluss an die Arbeit von Littlewood und Offord [Ann. of Math. (2) **49** (1948), 885–952; **50** (1949), 990–991; MR **10**, 692] untersucht Verf. die Verteilung der a-Stellen der meromorphen Funktionen $g(z, t)=b(z, t)/c(z, t)$, wo $b(z, t)=\sum b_n r_{2n}(t)z^n$ und $c(z, t)=\sum c_n r_{2n+1}(t)z^n$ ganze Funktionen in z von positiver endlicher Ordnung, und die $r_n(t)$ $(0\leqq t\leqq 1)$ Rademachersche Funktionen sind. Der Parameter wird als stochastische Variable aufgefasst. $v(z, r; t, a)$ bezeichnet die Anzahl der a-Stellen von $g(\zeta, t)$ im Kreis $|\zeta-z|<r$. Verfasser gibt eine Funktion $J(R, r, a)$ an, die ausser von den Grössen Rr und a nur noch von den Folgen $\{b_n\}$ und $\{c_n\}$ abhängig ist, und folgende Eigenschaft besitzt: Zu jedem $\varepsilon>0$ gibt es ein $R_0(\varepsilon)$ und auf dem Intervall $E=\{t|0\leqq t\leqq 1\}$ eine Menge A vom Mass $<\varepsilon$, die ausser ε nur von den b_n und c_n abhängt, sodass die Ungleichung $|v(z, r; t, a)-J(|z|, r, a)|<|z|^{\varepsilon}$ für alle $|z|$ mit $|z|\geqq R_0(\varepsilon)$, alle $t\in E-A$, alle a und alle r mit $|z|^{-\rho-2}<r<\frac{1}{2}|z|$ erfüllt ist.

Aus diesem Satz geht hervor, dass die a-Stellen der Funktionen $g(z, t)$, $t\in E-A$ sehr regelmässig verteilt sind. Diese Aussage hängt aber von den einzelnen a ab. Durch Abschätzung der Grösse $J(R, r, a)$ gewinnt der Verf. Aussagen, die gleichmässig gelten für alle a aus gewissen Gebieten. *A. Pfluger* (Zürich).

Lohwater, A. J. The reflection principle and the distribution of values of functions defined in a circle. Ann. Acad. Sci. Fenn. Ser. A. I. no. **229** (1956), 18 pp.

The main purpose of this paper is to unify some important results on functions of class (U) in the sense of Seidel [Trans. Amer. Math. Soc. **36** (1934), 201–226], by an elementary method which does not depend on the Poisson-Stieltjes-Herglotz integral representation. The author's method is based on the following: If $f(z)$ is a function of class (U) and if $f(z)\neq 0$ in $|z|<1$, then any component of the open set $H_n=\{z||z|<1;\ |f(z)|<1/n\}$, where n is a positive integer, is a simply connected domain bounded by a Jordan curve which has a non-empty intersection of linear measure zero with $|z|=1$. His method admits some applications to functions which are meromorphic in $|z|<1$ or are locally of a quasi-conformal character in $|z|<1$. *K. Noshiro* (Nagoya).

Gel'fer, S. A. The method of variations in the theory of p-valent functions. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 5(71), 60–66. (Russian)

Let $F_p^{g}(a_1, \cdots, a_m)$ denote the class of functions $f(\zeta)=\sum_{n=1}^{\infty}c_n\zeta^n$ which are regular and p-valent in $|\zeta|<1$, omit $a_1, \cdots, a_m$ in $|\zeta|<1$, and map $|\zeta|<1$ onto a domain D contained in a p-sheeted algebraic Riemann surface R of genus $\gamma\leqq g$, with $f(0)=0$ a fixed place on R. The author attacks the problem of maximizing $|c_1|$ in the class $F_p^{g}(a_1, \cdots, a_m)$. By means of the function $z=\Phi(Z)$ which maps (1) the entire complex plane, or (2) the punctured complex plane, or (3) the circle $|Z|<1$ onto the universal covering surface of R, he is able to use known variational techniques from the theory of univalent functions. The differential equations for the extremal functions involve the unknown $\Phi(Z)$ so that in general only qualitative results are obtained in this way. In the simple case of the family $F_p^{0}(a)$, the author gets sharp bounds for $|c_1|$, $|f(\zeta)|$ and $|f'(\zeta)|$, and these are contained in the results obtained by Hayman (Theorem 9) [J. Analyse Math. **1** (1951), 155–179; MR **13**, 545]. *A. W. Goodman.*

Zmorovič, V. A. On certain classes of analytic functions in a circular ring. Mat. Sb. N.S. **40(82)** (1956), 225–238. (Russian)

Let $B(q; 1)$ denote the class of functions $f(z)$ which are single-valued and regular in the ring $q<|z|<1$, and for

which

$$\text{(A)} \qquad \int_{-\pi}^{\pi} |\Re f(\rho e^{i\varphi})| d\varphi < L(f)$$

for $q<\rho<1$, where $L(f)$ is a constant which depends upon f but not on ρ.

The author obtains a structural formula in the form of a Stieltjes integral which characterizes this class of functions. This result is generalized (a) to the wider class of meromorphic functions with a finite number of poles, and (b) to the class of functions with a finite number of essential singularities, where now in (A) ρ is restricted to $q<\rho<q+\delta$ and $1-\delta<\rho<1$ to avoid the singular points. Various applications and special cases are indicated.

A. W. Goodman (Lexington, Ky.).

Maksimov, Yu. D. Estimation of coefficients for certain classes of analytic functions. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 507–510. (Russian)

Let $S_\pi(p, q)$ denote the class of functions

$$F(z)=z^q(1+a_1z+a_2z^2+\cdots)$$

which are regular in $|z|<1$ and such for each $F(z)$, there is a $\rho<1$ with the property that for $z=re^{i\theta}$

$$\text{(a)} \int_{\theta_1}^{\theta_2} \Re\left\{\frac{zF'(z)}{F(z)}\right\} d\theta > -\pi, \quad \text{(b)} \int_{-\pi}^{\pi} \Re\left\{\frac{zF'(z)}{F(z)}\right\} d\theta = 2p\pi$$

for $\rho<r<1$, and $\theta_1<\theta_2$. Here $1\leq q\leq p$.

A similar definition is stated for a class $C_\pi(p, q)$ of functions in such a way that, $F(z)\in S_\pi(p, q)$ if and only if $F(z)=zf'(z)/q$ where $f(z)\in C_\pi(p, q)$. The author obtains the sharp upper bounds for all the coefficients, for functions in the classes $C_\pi(p, q)$ and $S_\pi(p, q)$. If the function is assumed further to have m-fold symmetry about the origin, the author again finds sharp upper bounds for all the coefficients for functions in these subclasses of $C_\pi(1, 1)$ and $S_\pi(1, 1)$.

A. W. Goodman.

Bourion, Georges. Sur la relation de récurrence de Faber. Bull. Sci. Math. (2) **80** (1956), 73–76.

Let

$$w=\varphi(z)=z+b_0+\frac{b_1}{z}+\frac{b_2}{z^2}+\cdots,$$

$$z=\psi(w)=w+a_0+\frac{a_1}{w}+\frac{a_2}{w^2}+\cdots$$

be respectively the mapping function of the complement of a closed connected set E of the complex plane onto the region $|w|>r_0$ and its inverse. The corresponding nth Faber polynomial is defined as the polynomial term in the expansion of $[\varphi(z)]^n$, so that

$$P_n(z)=w^n+\frac{\alpha_{1,n}}{w}+\cdots.$$

Faber asserted [J. Reine Angew. Math. **150** (1920), 79–106] that

$$P_{n+1}(z)=zP_n(z)-A_0P_n(z) -A_1P_{n-1}(z)-\cdots-A_{n-1}P_1(z)-(n+1)A_n,$$

but only proved that the constant term appearing on the right-hand side is $-(A_n+\alpha_{1,n})$. The author shows that $\alpha_{1,n}=nA_n$ by residue calculus.

A. B. Novikoff.

Stein, Karl. Analytische Zerlegungen komplexer Räume. Math. Ann. **132** (1956), 63–93.

Die analytischen Zerlegungen tretenzu erst im Anschlusz an die analytischen Projektionen auf, die von K. Stein und K. Koch eingeführt wurden. In einem Gebiete G_z^n des Raumes C_z^n der n komplexen Veränderlichen $z_1, \cdots, z_n$ sei eine dort holomorphe Funktion f gegeben. Wenn Φ die holomorphe Abbildung von G_z^n auf die durch f definierte analytische Projektion Y_w^1 und φ die Projektion von Y_w^1 in den schlichten C_w^1 bedeutet, so ist $f=\varphi\circ\Phi$. Ist g irgendeine von f in G_z^n abhängige holomorphe Funktion, so gibt es ein auf Y_w^1 holomorphes ψ, sodasz $g=\psi\circ\Psi$. Zum Nachweis der Existenz der analytischen Projektion Y_w^1 wird von den 2 zulegungen $Z(f)$ und Z' von G_z^n ausgegangen, deren Elemente die Faserungen von f — d.h. die durch $f=$const in G_z^n gegebenen Komponenten dieser Faserungen — sind. (Im ersten Falle sind die Komponenten dieser Faserung nicht notwendig zusammenhängend; im zweiten Falle sind sie stets zusammenhängend). Jede von f in G_z^n abhängige, holomorphe, nicht konstante Funktion g führt zu einer Zerlegung $Z(g)$, die im allgemeinen von $Z(f)$ verschieden ist. Z' hängt jedoch nicht von den Wahl von g ab. $\bar{Z}$ sei die Durchschnitt aller $Z(g)$. Der Quotientenraum $G_w^n/\bar{Z}$ führt durch Aufprägung einer komplexen Struktur zu Y^1 (der abstrakten Riemannschen Fläche von Y_w^1).

Verf. wirft in der vorliegenden Arbeit die Frage auf, ob noch analoges gilt, wenn statt von einem Gebiete des C_z^n von einem beliebigen komplexen Raum X ausgegangen wird und an Stelle der Funktion f eine holomorphe Abbildung F von X in einen weiteren komplexen Raum X_1 auftritt.

Hier die Hauptresultate! X sei eine komplexe Mannigfaltigkeit und F eine holomorphe Abbildung von X in X_1. Die zusammenhängenden Komponenten der Fasermengen von F seien samtlich kompakt. Dann ist die durch F bestimmte einfache Zerlegung $Z'(F)$ von X eine normale analytische Zerlegung. Die natürliche Abbildung von X auf den komplexen Zahlenraum $X/Z'(F)$ ist eine eigentliche holomorphe Abbildung. — Dabei heiszt eine Zerlegung Z des komplexen Raumes X analytisch, wenn in der Menge M der Elemente von Z die Struktur eines komplexen Raumes so eingeführt werden kann, dasz die natürliche Abbildung von X auf M holomorph ist. Eine analytische Zerlegung heiszt normal, wenn es auf dem Quotientenraum X/Z genau eine Z zugeordnete komplexe Struktur gibt.

Die obige Aussage bleibt richtig, wenn X ein zusammenhängender n-dimensionaler komplexer Raum im Sinne von H. Cartan ist und F als holomorphe Abbildung vom globalen Range n oder $n-1$ vorausgesetzt wird.

H. Behnke (Münster).

See also: Wright, p. 635; Šilov, p. 657; Kohn and Spencer, p. 658; Rudin, p. 661; Froyd, p. 688; Conroy, p. 689.

Functions with Particular Properties

Nicolescu, Miron. Propriétés de moyenne des fonctions harmoniques bornées dans un demi-plan ou dans un angle droit. Rev. Math. Pures Appl. **1** (1956), no. 2, 43–50.

Certain elementary properties are derived for mean-values taken over the lines $y=$const of functions harmonic and bounded on the half-plane $y>0$. (Similar results are given for the region $x>0$, $y>0$.)

M. G. Arsove.

Tsuji, Masatsugu. Solution of Neumann's problem without use of integral equation. Comment. Math. Univ. St. Paul. **5** (1956), 123–127.

The author completes his treatment of the Neumann for a domain of a compact Riemann surface by filling a gap in the proof of an earlier paper [J. Math. Soc. Japan **6** (1954), 122–128; MR **16**, 349]. *F. W. Gehring.*

Gehring, F. W. On the Dirichlet problem. Michigan Math. J. **3** (1955–1956), 201.

L'auteur étend un théorème de W. Kaplan [même J. **3** (1955), 43–52; MR **17**, 31] et donne une démonstration du tout en quelques lignes. Etant donné $f(\theta)$ mesurable, p.p. finie, périodique (de période 2π), il existe dans le cercle unité une fonction harmonique qui, pour presque tous les θ, admet une limite (non tangentielle quelconque) $f(\theta)$. Il suffit d'introduire l'intégrale de Poisson $U_{r,\theta}$ relative à $F(\theta)$ continue de dérivée $f(\theta)$ p.p., et de considérer $\partial U/\partial\theta$. *M. Brelot* (Paris).

Tsuji, Masatsugu. A simple proof of a theorem of Erdös and Gillis on the transfinite diameter. Comment. Math. Univ. St. Paul. **5** (1956), 115–121.

The theorem of Erdös and Gillis [J. London Math. Soc. **12** (1937), 185–192] referred to is the following: If $\lambda(E)$ denotes the logarithmic measure of a closed and bounded set E in the plane and $\gamma(E)$ its logarithmic capacity, then a finite logarithmic measure implies a zero logarithmic capacity. The logarithmic measure is defined as the limit of $h(\rho)$ as $\rho\to 0$, where $h(\rho)=\inf\sum(\log 1/d_i)^{-1}$, where the d_i are the diameters of a finite collection of open circles covering E, $d_i<\rho$, and the lower bound is taken with respect to all methods of covering E. The logarithmic capacity is the maximum charge which E can bear, whose logarithmic potential remains $\leqq 1$ on E. *J. W. Green.*

★**Huber, Alfred. Some results on generalized axially symmetric potentials.** Proceedings of the conference on differential equations (dedicated to A. Weinstein), pp. 147–155. University of Maryland Book Store, College Park, Md., 1956.

The elliptic partial differential equation

$$L_k[u]\equiv\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2}+\frac{k}{x_n}\frac{\partial u}{\partial x_n}=0\quad(k=\text{real const})$$

is discussed with regard to (1) the Dirichlet problem for the operator L_k, (2) properties of mean-values of u, and (3) integral representations of u on the half-space $x_n>0$. For integral values of k the Weinstein correspondence principle reduces the study of u to the study of harmonic functions in $(n+k)$- or $(n+2-k)$-dimensional space, and this approach is used to obtain (2) and (3). In connection with (1) the author collects some of the known results for general values of k and furnishes certain extensions of his own. *M. G. Arsove* (Seattle, Wash.).

Dugué, Daniel. Résultats sur les fonctions absolument monotones et applications à l'arithmétique des fonctions de type positif. C. R. Acad. Sci. Paris **244** (1957), 715–717.

The author announces several results, the basic one being that if a product of positive powers of absolutely monotonic functions is analytic in a circle with center at 0 then each of the individual functions is analytic at least in the same circle. Further, if a product of positive powers of functions of positive type is analytic in a strip containing the real axis, each of the individual functions is analytic at least in the same strip. This leads to a theorem on characteristic functions of distributions and ultimately to the result that if a product of powers of characteristic functions is of the form $\exp\{\lambda(e^{-z}-1)\}$ then each characteristic function is of the form $\exp\{\lambda(e^{tz}-1)+i\mu z\}$.

R. P. Boas, Jr. (Evanston, Ill.).

See also: Marcus and Thompson, p. 634; Savage, p. 647; Kennedy, p. 647; Magenes, p. 655.

Special Functions

Maximon, Leonard C. On the representation of indefinite integrals containing Bessel functions by simple Neumann series. Proc. Amer. Math. Soc. **7** (1956), 1054–1062.

Starting with a series

$$(1)\qquad S_\nu[f(t),g(t)]=\sum_{m=0}^{\infty}[f(t)]^{m+\nu}J_{m+\nu}[g(t)],$$

the author derives a differential equation for S_ν. Solving for S_ν yields the sum of two integrals containing $J_\nu(g)$ and $J_{\nu-1}(g)$ in their integrands. If f and g are related by $\alpha fg+\beta g/f=\gamma$ (α, β, γ constants) and under the restriction that the integrals tend to zero for $\nu\to\infty$, one such integral is shown to be equal to the difference of two series of type (1). Three examples illustrate the paper. The method seems to be quite useful for the numerical computation of certain integrals containing a Bessel function in the integrand. *D. J. Hofsommer* (Amsterdam).

Ma, Min-Yuan; et Brepson, Roger. Sur le calcul des expressions $J_\nu(Z)Y_{\nu+p}(z)\pm Y_\nu(Z)J_{\nu+p}(z)$. C. R. Acad. Sci. Paris **244** (1957), 548–550.

The authors obtain new and improved asymptotic expansions for the function in the title. *E. Pinney.*

Fettis, Henry E. On the evaluation of two functions occurring in Maslen and Moore's theory of strong transverse waves in a circular cylinder. J. Aero. Sci. **24** (1957), 64–65.

These formulae are derived:

$$\int_0^\pi[J_{2n}(2\alpha\cos\theta)-J_{2n}(2\alpha)]\csc^2\theta\,d\theta=-4\pi\sum_{r=1}^{\infty}rJ_{n+r}(\alpha)J_{n-r}(\alpha),$$

$$\int_0^\pi[J_0(2\alpha\cos\theta)-J_0(2\alpha)]\cos(2n\theta)\csc^2\theta\,d\theta=(-1)^{n-1}4\pi\sum_{r=1}^{\infty}(-1)^rrJ_{n+r}^2(\alpha).$$

The Neumann series for $J_{2n}(2\alpha\cos\theta)$ is substituted in the integrals thereby reducing the problem to an evaluation of $\int_0^\pi\sin^2 r\theta\csc^2\theta\cos 2n\theta\,d\theta$. The rather involved evaluation of this integral in the appendix could have been replaced by the observation that the integral represents a multiple of the Fourier coefficients in the known cosine expansion of the function $\sin^2 r\theta\csc^2\theta$ [Magnus and Oberhettinger, Formeln und Sätze für die speziellen Funktionen der mathematischen Physik, Springer, Berlin, 1948, p. 218; MR **10**, 38]. *R. G. Langebartel.*

See also: Kumar, p. 652; Davis, Scott, Springer and Resch, p. 653; Miller and Freund, p. 681; Luchak, p. 707.

Sequences, Series, Summability

Benedetti, Carlo. Di un massimo dell'indice quadratico di oscillazione. Metron **18** (1956), no. 1–2, 163–180.

Let $(a_i)=(a_1\leq a_2\leq\cdots\leq a_n)$ be a given non-decreasing sequence of real numbers ≥ 0, and let (b_i) denote a permutation of (a_i). The author determines the maximum of

$$\sqrt{\Big(\sum_{i=1}^{n-1}(b_i-b_{i-1})^2/(n-1)\Big)}$$

for all $n!$ permutations (b_i) and finds the extremizing permutation. *Z. W. Birnbaum* (Seattle, Wash.).

Peyerimhoff, Alexander. Über Summierbarkeitsfaktoren und verwandte Fragen bei Cesàroverfahren. II. Acad. Serbe Sci. Publ. Inst. Math. **10** (1956), 1–18.

This is the second part of author's paper [same Publ. **8** (1955), 139–156; MR **17**, 1076]. Using a lemma which expresses differences of type $\Delta^\alpha\varepsilon_n\delta_n$ (with bounded ε_n, δ_n and not integral $\alpha\geq 0$) by means of a finite sum of products of differences $\Delta^\lambda\varepsilon_n$, $\Delta^\mu\delta_n$ (λ, μ integers) and a remainder, results of the following type are obtained. Let α, $\beta\geq 0$ and $\varepsilon_n=O(1)$. The relation $\Delta^\alpha\varepsilon_n\delta_n=O(n^{-\alpha-1})$ holds for all δ_n with $\delta_n\to 0$, $\Delta^\beta\delta_n=O(n^{-\beta-1})$ if and only if $\Delta^\alpha(\varepsilon_n/n)=O(n^{-\alpha-1})$ and in case $\beta\leq\alpha$ in addition $\Delta^\alpha(\varepsilon_n/n)=O(n^{-\alpha-1})$. These results serve to obtain characterizations of the summability factors of class (C_α, C_β) due to Bosanquet [Proc. London Math. Soc. (2) **50** (1949), 482–496; MR **10**, 368], of class $(|C_\alpha|, |C_\beta|)$ due to Andersen [Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 1–4; MR **16**, 464], Chow [J. London Math. Soc. **29** (1954), 459–476; MR **16**, 464] and the author [Math. Z. **59** (1954), 417–424; MR **15**, 617] and of class $(C_\alpha, |C_\beta|)$.

G. G. Lorentz (Ann Arbor, Mich.).

Peyerimhoff, Alexander. Über ein Lemma von Herrn H. C. Chow. J. London Math. Soc. **32** (1957), 33–36.

H. C. Chow [same J. **29** (1954), 459–476; MR **16**, 464] showed that if the transformation $y_n=\sum_{k=0}^n a_{nk}x_k$ is such that $\sum|y_n|<\infty$ whenever $x_k=O(1)$, then $\sum|a_{nn}|<\infty$. Let P (or P_1) be the class of matrices a_{nk} of complex elements which are such that the series in $y_n=\sum_{k=0}^\infty a_{nk}x_k$ converges for each $n=0, 1, 2, \cdots$ and $\sum|y_n|<\infty$ whenever $x_0, x_1, \cdots$ is a complex sequence for which $x_k=O(1)$ (or $x_k=o(1)$). The question whether $a_{nk}\in P$ implies $\sum_{n,k=0}^\infty|a_{nk}|<\infty$ was left open by Chow. The paper under review gives several formulations of necessary and sufficient conditions that $a_{nk}\in P$, shows that $P=P_1$, and gives an example to show that the hypothesis that $a_{nk}\in P$ does not imply that $\sum_{n,k=0}^\infty|a_{nk}|<\infty$. One necessary and sufficient condition that $a_{nk}\in P$ is existence of a constant C such that $|\sum_{n\in N,k\in K}a_{nk}|<C$ whenever N and K are finite sets of positive integers. The author notes in an appendix to his paper that this condition was given by Zeller [Rend. Mat. e Appl (5) **12** (1953), 340–346; MR **15**, 618]. *R. P. Agnew.*

See also: Wright, p. 635; Wirsing, p. 642; Meinardus, p. 642; Hodges, p. 643; Vijayaraghavan and Padmavally, p. 646; Maximon, p. 650; Rozovskiĭ, p. 688.

Approximations, Orthogonal Functions

Goormaghtigh, R. Construction de formules approchées. Mathesis **65** (1956), 505–508.

Close approximations to several rational functions of x, $\cos\alpha x$, and $\sin\beta x$ for specific α and β are given. Let a and b be the semi-axes of an ellipse with eccentricity e and perimeter L. Then

$$L=\frac{\pi}{8}\Big[9(a+b)-5(ab)^{1/2}+3\Big(\frac{a^2+b^2}{2}\Big)^{1/2}\Big]-7.2^{-30}e^{12}\cdots.$$

The neglected terms are of the form a_ne^{2n}, $n>6$, $a_n<0$.

N. D. Kazarinoff (Ann Arbor, Mich.).

Mergelyan, S. N. On Bernstein's approximation problem. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 25–28. (Russian)

The author announces another solution of S. Bernstein's problem about weighted polynomial approximation on the real axis, and several related results. [Proofs, and references to earlier work not mentioned in this note, appear in the author's paper in Uspehi Mat. Nauk **11**, no. 5 (71), 107–152 (1956).] Let $h(x)$ be measurable, with $0\leq h(x)\leq 1$; let E_h be the set where $h(x)>0$; let C_h be the class of functions $f(x)$ continuous on the real axis with $\lim h(x)f(x)=0$; let H be the class of $h(x)$ for which the class of polynomials is complete in C_h with the distance between f and P defined by $\sup\{h(x)|f(x)-P(x)|\}$. Let $\mathfrak{M}_h$ denote the set of polynomials such that $h(x)|P(x)|\leq 1$, and let $M_h(z)=\sup|P(z)|$ with $P\in\mathfrak{M}_h$. With $h(x)$ the author associates the regularized function $h^*(x)=(1+|x|)/M_k(x)$, where $k(x)=h(x)/(1+|x|)$. Then (1) $h\in H$ if and only if

$$\int_{-\infty}^{\infty}(1+x^2)^{-1}\log h^*(x)dx$$

diverges. (2) Also, $h\in H$ if and only if, for every positive R and N, there exists a polynomial P such that

$$h(x)|P(x)|\leq 1+|x|$$

on $(-\infty,\infty)$, and $\max_{(-R,R)}|P'(x)|>N$. (3) If $h(x)$ is not in H, then every f which admits weighted polynomial approximation with weight h coincides on E_h with an entire function of at most order 1, minimal type. (4) Let $n(r)$ denote the number of points of E_h for which $|x|<r$. If $\limsup r^{-1}n(r)>0$, then for every positive s the functions $h(x)$ and $(1+|x|)^sh(x)$ both belong or both fail to belong to H. (5) If the condition in (4) holds and $h(x)\in H$, then the set of all polynomials not containing a given finite set of positive powers of x is complete in C_h. (6) Let $\omega_f(\delta)=\sup|f(x_1)-f(x_2)|$ for

$$|x_1-x_2|\leq\delta((1+|x_1|)(1+|x_2|).$$

Let p_n be the infimum of the values of polynomials in $\mathfrak{M}_k$, $k(x)=h(x)/(1+|x|)$, of degree at most n, at a given real point a. Then for every positive n there is a polynomial $P_n(x)$ of degree n such that

$$h(x)|f(x)-P_n(x)|<C\omega_f(1/\log p_n),$$

where C is independent of n. Finally (7) if the real axis is replaced by any closed nowhere dense set F in the plane, the set of polynomials is complete on F with weight h if and only if in every region complementary to F we have $M_k(z)=\infty$, with $k(z)=h(z)/(1+|z|)$.

R. P. Boas, Jr. (Evanston, Ill.).

See also: Ma et Brepson, p. 650; Collatz, p. 677.

Trigonometric Series and Integrals

Kahane, Jean-Pierre; et Salem, Raphaël. Sur les ensembles linéaires ne portant pas de pseudomesures. C. R. Acad. Sci. Paris **243** (1956), 1185–1187.

Let E be a closed set in $[0, 2\pi]$ and let $C=\{c_n\}_{-\infty}^{\infty}$ be a

bounded sequence such that $c_0x+\sum (in)^{-1}c_ne^{inx}$ represents a function constant on each interval contiguous to E. E is said to be without true pseudomeasure if any C is the sequence of Fourier-Stieltjes coefficients of a measure. The author establishes the existence of a perfect set E without true pseudomeasure. Various properties of sets without true pseudomeasure are given. In particular any function continuous on E can be prolonged to a function on $[0, 2\pi]$ having an absolutely convergent Fourier series.

P. Civin (Eugene, Oreg.).

Izumi, Shin-ichi. Fourier series. III. Wiener's problem. Proc. Japan Acad. **32** (1956), 527–528.

Let $s_n(x)$ be the nth partial sum of the Fourier series of $f(x)$ and let

$$\omega_p(t)=\sup_{|u|\leq t}\left\{\int_0^{2\pi}|f(x+u)-f(x)|^p dx\right\}^{1/p}.$$

The author shows that $\sum_{n=1}^{\infty}\omega_p(1/n)<\infty$ is sufficient for the almost everywhere convergence of

$$\sum_{n=1}^{\infty}|s_n(x)-f(x)|^p.$$

The result partially extends a result of Kinukawa [same Proc. **32** (1956), 86–89; MR **17**, 1079]. *P. Civin.*

Mohanty, R. Evaluation of a trigonometric integral. Proc. Amer. Math. Soc. **8** (1957), 107–110.

Let $f(t)$ be integrable on the real line and let x be a point such that $\psi(t, x)=f(x+t)-f(x-t)=O(t^{\alpha})$ as $t\to 0+$, $0<\alpha<1$. Let $a(u)$ and $b(u)$ be $\int_{-\infty}^{\infty}f(t)c(ut)dt$ with $c(t)$ equal to $\cos t$ and $\sin t$ respectively. Then

$$\int_0^{\infty}u^{\alpha}\{b(u)\cos xu-a(u)\sin xu\}du$$

is $(C, 1)$ summable to $\Gamma(1+\alpha)\cos(\alpha\pi/2)\int_{0+}^{\infty}\psi(t, x)t^{-1-\alpha}dt$ whenever the latter integral exists. *P. Civin.*

See also: Minasyan, p. 655; Croce, p. 697.

Integral Transforms

Macdonald, J. Ross; and Brachman, Malcolm K. Linear-system integral transform relations. Rev. Mod. Phys. **28** (1956), 393–422.

This is essentially a discussion of Laplace, Fourier, Hilbert, Kronig-Kramers, Stieltjes and Mellin transforms and their formal interrelationships, interpreted in terms of many kinds of physical problems and intended for physicists. The authors are careful to point out places where purely formal calculation can lead one astray; their solution in each case is to introduce an ad hoc device to salvage the calculations, and it seems that even the intuition of physicists is sometimes not sufficient guide. Apparently in some cases more attention to mathematical rigor would actually prevent physical mistakes. A table (the meanings of whose headings unfortunately have to be excavated from the body of the paper) lists, for 21 functions, what are essentially the real and imaginary parts on the imaginary axis, the cosine transform of one of these, the inverse Stieltjes transform, and the Mellin transform of this; since these are various transforms of each other and of other things, further transform pairs can be read off by reference to the rest of the paper. Quite a number of the individual transform pairs seem not to be available in the usual tables. The paper contains many other interesting formal relations and is also a useful reference for the meanings of technical terms used qy physicists. *R. P. Boas, Jr.* (Evanston, Ill.).

Kumar, Ram. Some recurrence relations of the generalised Hankel-transform. I. Ganita **5** (1954), 191–202 (1955).

Kumar, Ram. Some recurrence relations of the generalised Hankel transform. II. Ganita **6** (1955), 39–53 (1956).

The transform is question is

$$f(x)=2^{-\lambda}\int_0^{\infty}(xy)^{\lambda+\frac{1}{2}}J_{\lambda}^{\mu}(\tfrac{1}{4}x^2y^2)g(y)\,dy,$$

where

$$J_{\lambda}^{\mu}(x)=\sum_{r=0}^{\infty}(-x)^r/\{r!\Gamma(1+\lambda+\mu r)\}$$

[Agarwal, Ann. Soc. Sci. Bruxelles. Ser. I. **64** (1950), 164–168; MR **12**, 605]. Put

$$\psi_{\lambda,\mu,\nu}=\int_0^{\infty}(xy)^{\nu}J_{\lambda}^{\mu}(xy)g(y)\,dy.$$

The author establishes several recurrence relations for $\psi_{\lambda,\mu,\nu}$. He uses these to evaluate a number of definite integrals involving J_{λ}^{μ}, in particular integrals involving Bessel functions and Hermite polynomials, and also to obtain recurrence relations for special cases of the J_{λ}^{μ}.

R. P. Boas, Jr. (Evanston, Ill.).

See also: Wirsing, p. 642; Lakshmana, p. 644; Dugué, p. 650; Mainra, p. 657; Ramakrishnan et Srinivasan, p. 680; Žautykov, p. 691; Lane, p. 697; Kesten and Runnenburg, p. 708.

Ordinary Differential Equations

Wintner, Aurel. Addenda to the paper on Bôcher's theorem (vol. 76 (1954), pp. 183-190). Amer. J. Math. **78** (1956), 895–897.

If $(a_{ik})=a=a(t)$, where $0\leq t<\infty$, is an $n\times n$ matrix of continuous functions $a_{ik}(t)$, let $a\in S$ mean that for every constant vector c there is a solution $x(t)$ of $\dot{x}=a(t)x$ with $x(\infty)=c$.

In the paper cited in the title of this note [see MR **15**, 426] the author proved that if $a\in L$ (absolutely integrable on $0\leq t<\infty$), then $a\in S$. The condition $a\in R$, conditional integrability, is not sufficient for $a\in S$. Here the author points out that: Theorem. If $a_0(t)$ denotes the diagonal matrix consisting of the diagonal elements of $a(t)$, and $a_0\in R$, $a-a_0\in L$, then $a\in S$.

As a second item the author finds that if each of the three functions $f_k(t)=f(t)e^{2kti}$, $k=-1, 0, 1$, $i^2=-1$, satisfy $f_k\in R$ and $F_k\in L$, where $F_k(t)=f(t)\int_t^{\infty}f(s)e^{-2kti}ds$, then $\ddot{x}+[1+f(t)]x=0$ represents an adiabatic perturbation of the harmonic oscillator. *L. Markus.*

Pliś, A. Sets filled by asymptotic integrals of ordinary differential equations. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 749–752.

The author considers the system of $k+m$ real ordinary differential equations

$$\dot{x}=f(x, y, t),\ \dot{y}=g(x, y, t), \tag{1}$$

where $x=(x_1, \cdots, x_k)$, $y=(y_1, \cdots, y_m)$, $f=(f^1, \cdots, f^k)$, $g=(g^1, \cdots, g^m)$. It is assumed that f and g are continuous

in an open set S of $(k+m+1)$ space, and therein satisfy certain unusual inequalities. Also it is assumed that (1) has a unique solution for each initial point in S.

Let U be the set of points (X, Y, T) lying on solution curves of (1) which remain in S for $T \leqq t < \infty$ and which also satisfy a certain growth condition. The principal theorem states that U is the graph of a single-valued Lipschitz function $Y=\psi(X, T)$. *L. Markus.*

Gallo, Elisa. I sistemi [G] e il secondo teorema di Moutard. Univ. e Politec. Torino. Rend. Sem. Mat. **15** (1955–56), 329–342.

Differential equations of the type (G):

$$y'''=G(x, y, y')y''+H(x, y, y')y''^2,$$

were originally introduced and investigated by Kasner [Differentialgeometric aspects of dynamics, Amer. Math. Soc. Colloq. Publ., v. 3, New York, 1913]. Under certain general conditions, the author proves that through a general lineal element $E(x, y, y')$, there pass three integral curves which are hyperosculated by their osculating conics. Systems of differential equations of the type (G) are determined such that through any lineal element $E(x, y, y')$, there pass at most two (and also one) integral curves which are hyperosculated by their osculating conics. If the integral curves of a differential equation of the type (G) are not all conics, then it possesses at most three ∞^2 conics as integral solutions. Dynamical families of conics were orginally obtained by Darboux.

J. De Cicco (Chicago 1, Ill.).

Olech, C. On the asymptotic behaviour of the solutions of a system of ordinary non-linear differential equations. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 555–561.

By the use of the theory of Ważewski [Ann. Soc. Polon. Math. **20** (1947), 279–313; MR **10**, 122], the author obtains generalizations of known results on the asymptotic integration of perturbed linear systems with constant coefficients, say, $y'=Ay+\Phi(t, y)$ as $t\to\infty$ [cf. Hartman and Wintner, Amer. J. Math. **77** (1955), 692–724; Trans. Amer. Math. Soc. **81** (1956), 1–24; MR **17**, 485, 850]. The theorems are too long to state here. It can be mentioned that if $y=(y_1, \cdots, y_n)$ and if A is in a Jordan normal form, one type of generalization consists of replacing majorants for Φ involving $\sum |y_k|$ in hypotheses by majorants involving the larger quantities $\sum t^j|y_k|$, where $j=j_k$ are suitably chosen non-negative integers [in the linear case, cf. Hartman and Wintner, Amer. J. Math. **77** (1955), 45–86, 404; MR **16**, 590].

P. Hartman (Baltimore, Md.).

Richard, Ubaldo. Alcune proprietà e formule asintotiche per le soluzioni della equazione $[py']'-qy=0$. Univ. e Politec. Torino. Rend. Sem. Mat. **15** (1955–56), 175–200.

Der Verfasser studiert zuerst die asymptotischen Eigenschaften des Integrals $y_1(x)$ der Gleichung

$$[py']'-qy=0, \tag{1}$$

welches die Anfangsbedingungen

$$y_1(x_0)=y_0>0,\ y_1'(x_0)=y_0'>0$$

erfüllt. Dabei setzt er voraus, dass p und q positive und absolut stetige Funktionen in jedem abgeschlossenen Teilinterval von (a, ∞) sind. Er beweist ausser anderem, dass für $\lim_{x\to\infty} y_1(x)=\infty$ notwendig und hinreichend $I(\infty)=\infty$ ist, wobei

$$I(x)=\int_{x_0}^{x}\frac{du}{p(u)}\int_{x_0}^{u}q(t)dt.$$

Weiter beweist er bekannte asymptotische Formeln

$$y_1(x)\sim k(pq)^{-\frac{1}{4}}e^{\xi},p(x)y_1'(x)\sim k(pq)^{\frac{1}{4}}e^{\xi},\ \xi=\int_{x_0}^{x}\sqrt{(q(t)/p(t))}\,dt,$$

unter der Voraussetzung $\int^{\infty}|\omega|^{\rho}d\xi<\infty$, $1\geqq\rho\leqq 2$, wobei $\omega(x)=\frac{1}{2}p[(pq)^{-\frac{1}{2}}]'$. Dies ist ein bekanntes Resultat nur im Fall $\rho=1$, wie auch der Verfasser bemerkt. Endlich untersucht er die Integrale von (1) unter der Voraussetzung der Monotonie von pq. Es handelt sich um eine Verallgemeinerung der Resultate von Hartman und Wintner [Amer. J. Math. **76** (1954), 207–219; MR **15**, 527]. *M. Zlámal* (Brno).

★ **Diliberto, S. P.; and Hufford, G. Perturbation theorems for non-linear ordinary differential equations.** Contributions to the theory of nonlinear oscillations, vol. 3, pp. 207–236. Annals of Mathematics Studies, no. 36. Princeton University Press, Princeton, N. J., 1956. $4.00.

The authors study vector systems

$$\frac{dx}{dt}=f(x), \tag{1}$$

and a perturbed system

$$\frac{dx}{dt}=f(x)+\varepsilon R(x, t, \varepsilon), \tag{2}$$

where the unperturbed system has a family of periodic solutions (degenerate case). By the introduction of normal coordinates in the neighborhood of a periodic integral of (1), a periodic solution becomes a repeating solution. The results include the existence of a family of repeating solutions of (2), in normal coordinates, in a neighborhood of $\varepsilon=0$ when $R(x, t, \varepsilon)$ is of class C^2, and of the same period as the unperturbed solution integrals. This includes a result of Poincaré as a special case.

Other results generalize a theorem of Krylov and Bogoliubov to a system with k-fold degeneracy, with a periodic perturbation, in which the unperturbed solutions are asymptotically stable, obtaining a family of asymptotically stable tori (with the periods of R and of the unperturbed solution), which tend to zero with ε.

G. Latta (Palo Alto, Calif.).

★ **Davis, Harold T.; Scott, Walter; Springer, George; and Resch, Daniel. Studies in differential equations.** Northwestern University Press, Evanston, Illinois, 1956. vi+114 pp. $1.75.

This study consists of five papers on differential equations: (1) Studies relating to a non-linear differential equation of second order, with special reference to the first and second transcendent of Painlevé, by H. T. Davis; (2) Baecklund Transformations which leave partial differential equations invariant, by G. Springer; (3) Linear Baecklund Transformations, by W. T. Scott; (4) Note on the hodograph transformation in three dimensions, by H. T. Davis; (5) Some Baecklund transformations of partial differential equations of second order, by D. Resch.

In paper (1) the first Painlevé transcendent, satisfying $y''=6y^2+\lambda x$, $y(0)=1$, $y'(0)=0$ and the second Painlevé transcendent satisfying $y''=2y^3+xy+\lambda$, $y(0)=1$, $y(0)=0$

are studied and tabulated for $-1\leq x\leq 1$ with $\Delta x=0.01$ and parameter values $\lambda=0, 1, 2, 3, 4, 5$. In the study of the first equation the author analyses a new transcendent $S(v, k; \lambda)$ which, for $\lambda=0$, reduces to the Jacobi elliptic function $\mathrm{sn}(v, k)$.

In paper (2) the author considers first order partial differential systems $\zeta_u=H\zeta_v$ and $\zeta_u'=H'\zeta_v'$, where $\zeta=\begin{pmatrix}\xi\\ \eta\end{pmatrix}$, $\zeta'=\begin{pmatrix}\zeta'\\ \eta'\end{pmatrix}$ and H, H' are matrices of real functions of the two real variables (u, v). These two systems are to be related by a Baecklund transformation

$$\zeta_u'=W\zeta_u+A\zeta+C\zeta',\ \zeta_v'=W\zeta_v+B\zeta+D\zeta',$$

where A, B, C, D, W are 2×2 matrices depending on (u, v) and satisfying certain conditions. The author investigates such Baecklund transformations which carry $\zeta_u=H\zeta_v$ into itself.

In paper (3) the author constructs a general theory on the linkage of partial differential equations by linear and reversible Baecklund transformations. The restriction on the coefficients of a Baecklund transformation, that it produce the desired linkage of second order systems, is that a certain integrability condition is satisfied. A particular result is that two linked second order linear systems, in n independent variables, are necessarily of the same type (regarding signature).

In paper (4) the author notes certain quasi-linear partial differential equations in three independent variables which become linear equations under the hodograph transformation (interchanging dependent and independent variables).

The last paper (5) is closely related to (2) and (3) and investigates linear Baecklund transformations. By a Baecklund transformation the equations

$$z_{x_1x_1}+z_{x_2x_2}+z_{x_3x_3}+z_{x_4x_4}+\frac{1}{x_3}(z_{x_1}+z_{x_3})-\frac{4z}{(x_1-x_2)^2}=0,$$

$$z_{x_1x_1}+z_{x_2x_2}+z_{x_3x_3}+z_{x_4x_4}=0$$

are linked and, furthermore, it is shown that this cannot be achieved by simpler transformations on the dependent and independent variables. *L. Markus.*

Bernstein, B.; and Truesdell, C. The solution of linear differential systems by quadratures. J. Reine Angew. Math. **197** (1957), 104–111.

Let $A(t)$ denote the $n\times n$ matrix $(A_{ij}(t))$. The authors consider the general linear system of the nth order, which they write as $\dot{x}=A(t)x+b(t)$. [The functions $A_{ij}(t)$ and $b_i(t)$ are continuous functions of the independent variable t.] The main object of the paper is to find necessary and sufficient conditions for the existence of a non-singular matrix T such that its entries be constant and $B=TAT^{-1}$ be subdiagonal (i.e., $B_{ij}=0$ if $j>i$). The existence of T clearly reduces the solution of the original system to the solution of $\dot{y}=By+Tb$, where $y=Tx$. The functions y_i may be calculated by quadratures because of the subdiagonal character of B. *A. Edrei* (Syracuse N.Y.).

Putnam, C. R. Note on a one dimensional non-conservative system Amer. Math. Monthly **64** (1957), 32–33.

The author studies the system of two linear differential equations: $\dot{x}=f(t)y$, $\dot{y}=-f(t)x$, in which $f(t)$ is continuous and periodic with the period $p\neq 0$ in $-\infty<t<+\infty$. Only non-trivial solutions, that is, $[x(t)]^2+[y(t)]^2$, is not identically zero, will be considered. Every such solution $x=x(t)$, $y=y(t)$, is almost periodic. The following theorem is established. If $f(t)$ possesses a single (proper, relative) maximum and a single (proper, relative) minimum over a period p and if $x=x(t)$, $y=y(t)$ is any periodic non-trivial solution having p as a period, then either (a) at least one of the functions $x(t)$ or $y(t)$ has a non-zero constant term in its Fourier expansion or (b) the curve:

$$u=u(t)=\int_0^t x(s)ds,\ v=v(t)=\int_0^t y(s)ds,$$

is a closed curve of class two consisting of exactly two loops. *J. De Cicco* (Chicago, Ill.).

★ **Hufford, George. Banach spaces and the perturbation of ordinary differential equations.** Contributions to the theory of nonlinear oscillations, vol. 3, pp. 173–195. Annals of Mathematics Studies, no. 36. Princeton University Press, Princeton, N. J., 1956. $4.00.

The paper is about a theorem concerning the existence of a periodic solution of an autonomous system of differential equations with a periodic perturbation. In a note added in press, the author states that his proof is false but that the theorem itself is a special case of a result of W. T. Kyner [same Contributions, pp. 197–205; MR **18**, 408]. *J. Cronin* (New York, N.Y.).

Krasovskiĭ, N. N. On the theory of Lyapunov's second method in studying stability of motion. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 460–463. (Russian)

Consider $\dot{x}=f(t, x)$ where x is an n-vector, f is of class C^1 with $(\partial f/\partial x)$ bounded on R: $\|x\|\leq\rho$, $t\geq 0$, and $f(t, 0)\equiv 0$. Let $V(t, x)$ be a scalar function, of class C^1, positive definite and satisfying $V'(t, x)\equiv\partial V/\partial t+\operatorname{grad} V\cdot f\leq 0$ on R. The author states without proof: Theorem 1. A necessary and sufficient condition for $x=0$ to be both uniformly stable and equi-asymptotically stable is the existence of a V such that $V\to 0$ with $\|x\|$ uniformly in t and $\int_0^\infty\phi(t,\varepsilon)dt=-\infty$ for every $\varepsilon>0$, where $\phi(t, \varepsilon)=\sup V'(t, x)$ on $\varepsilon\leq\|x\|\leq\delta_0$. Theorem 2. A necessary and sufficient condition for $x=0$ to be asymptotically stable is the existence of a V such that $\int_0^\infty\psi(t, \varepsilon)dt=-\infty$ for every $\varepsilon>0$, where $\psi(t, \varepsilon)$ is a continuous function of t satisfying $\psi(t, \varepsilon)\geq\sup V'(t, x)$ on $\varepsilon\leq V$, $\|x\|\leq\delta_0$. A theorem on stability in the first approximation for equations with time delays and two theorems on stability in abstract spaces are also given. *H. A. Antosiewicz* (Washington, D.C.).

★ **Haas, Violet B. On a non-linear differential equation containing a small parameter.** Contributions to the theory of nonlinear oscillations, vol. 3, pp. 57–83. Annals of Mathematics Studies, no. 36. Princeton University Press, Princeton, N. J., 1956. $4.00.

The author considers a singular perturbation problem for a non-linear differential equation

$$(*)\qquad \varepsilon\frac{d^2x}{dt^2}+f\left(t, x, \frac{dx}{dt}; \varepsilon\right)=0$$

imposing conditions on $f(t, x, y; \varepsilon)$ so as to achieve a solution of (*), stable in an interval, and unstable in an adjoining interval, as $\varepsilon\to 0$. The list of conditions on $f(t, x, y; \varepsilon)$ unfortunately contains a number of errors and misprints and the author wishes to point out that in Section 2, (ix) should be deleted, (x) should read "$yf_{yy}>0$", and to (1) should be added "$f(t, x, y; \varepsilon)$ is odd in x and y in D".

Apparently this paper should generalize the case $f=x-4y/(3+y^4)$, but the conditions given do not work for this example. *G. Latta* (Palo Alto, Calif.).

See also: Linnik, p. 639; Grabar', p. 662; Fogagnolo Massaglia, p. 685; Kolomenskiĭ, p. 698; Bouwkamp and Casimir, p. 700; Duffin, p. 700.

Partial Differential Equations

Magenes, Enrico. Su alcune recenti impostazioni dei problemi al contorno, in particolare misti, per le equazioni lineari ellittiche del secondo ordine. Ann. Scuola Norm. Sup. Pisa (3) **10** (1956), 75–84.

Sia T un dominio piano delimitato da una curva di classe 2 e sia $\{\mathscr{F}_\tau\}$ il sistema di curve parallele a $\mathscr{F}T$, ottenute prendendo per ogni punto M di $\mathscr{F}T$ sulla normale interna un punto M a distanza τ da M. Se $F(P, Q)$ è una soluzione fondamentale di una equazione ellittica (in cui il coefficiente dell'incognita sia negativo) e $z(Q)$ è una funzione di quadrato sommabile su T, il potenziale generalizzato di semplice strato

$$v(P)=\int_{\mathscr{F}T}F(P, Q)z(Q)dS_Q$$

ammette su $\mathscr{F}_\tau$ una derivata conormale che per $\tau\to 0$ converge in media verso una funzione di quadrato sommabile su $\mathscr{F}T$. Questo interessante risultato permette all'A. di dimostrare che la classe funzionale in cui egli ha recentemente studiato il problema misto relativo alle equazioni ellittiche [gli stessi Ann. (3) **8** (1954), 93–120; **9** (1955), 161–200; MR **16**, 365; **18**, 45] coincide con un'altra che è suscettibile di una definizione assai più naturale ed espressiva. Analogo confronto si può poi istituire fra le classi funzionali considerate da L. Amerio [Rend. Mat. e Appl. (5) **5** (1946), 84–120; MR **9**, 37] e G. Cimmino [Convegno Internazionale sulle Equazioni Lineari alle Derivate Parziali, Trieste, 1954, Edizioni Cremonese, Roma, 1955, pp. 76–85; MR **17**, 604] per lo studio del problema di Neumann.

Il lavoro contiene infine un'estensione di un precedente risultato di L. Myrberg [Ann. Acad. Sci. Fenn. Ser. A. I. no. **103** (1951); MR **13**, 743] relativo al problema misto per l'equazione di Laplace. Secondo tale estensione la soluzione del problema esiste ed è unica (in un'opportuna classe funzionale) anche se, per quanto riguarda la condizione al contorno relativa alla derivata normale, si richiede soltanto che essa sia soddisfatta a meno di un insieme di capacità nulla. *C. Miranda* (Napoli).

Cinquini-Cibrario, Maria. Moderne ricerche sulle equazioni a derivate parziali del primo ordine. Univ. e Politec. Torino. Rend. Sem. Mat. **15** (1955–56), 5–26.

Una conferenza aggiornata sugli ultimi risultati relativi alle equazioni alle derivate parziali del primo ordine dal punto di vista della teoria delle funzioni di variabile reale. *G. Scorza-Dragoni* (Padova).

Manfredi, Bianca. Su la genesi dei pluriderivatori. Boll. Un. Mat. Ital. (3) **11** (1956), 538–543.

Minasyan, R. S. On a solution of the problem of Dirichlet for certain polygonal regions. Akad. Nauk Armyan. SSR. Dokl. **22** (1956), no. 5, 193–202. (Russian. Armenian summary)

Computation, by Fourier series methods, of the solution of the Dirichlet problem for a region shaped like the cross-section of an unsymmetrical I-beam.

M. G. Arsove (Seattle, Wash.).

Skorobogat'ko, V. Ya. Theorem on differential inequalities for an elliptic equation. Ukrain. Mat. Ž. **8** (1956), 335–338. (Russian)

Given the differential operator:

$$Lu=a_{11}\frac{\partial^2 u}{\partial x_1^2}+a_{12}\frac{\partial^2 u}{\partial x_1\partial x_2}+a_{21}\frac{\partial^2 u}{\partial x_1\partial x_2}+a_{22}\frac{\partial^2 u}{\partial x_2^2}+2b_1\frac{\partial u}{\partial x_1}+2b_2\frac{\partial u}{\partial x_2}+cu$$

in which the a_{ij} have continuous second derivatives, the b_i have continuous first derivatives, and function c is continuous in a region D with boundary S of Liapounoff type; let B_1 and B_2 be functions continuous and having piecewise continuous first derivatives in D. Let

$$A_i=\frac{1}{2}\sum_{k=1}^{2}\frac{\partial a_{ki}}{\partial x_k}-b_i+B_i \text{ and } R=\frac{\partial B_1}{\partial x_1}+\frac{\partial B_2}{\partial x_2}-c.$$

Theorem: Suppose there exist B_1 and B_2 such that for all points in $D+S$,

$$(*)\qquad \begin{vmatrix} a_{11} & a_{12} & A_1 \\ a_{21} & a_{22} & A_2 \\ A_1 & A_2 & R \end{vmatrix}>0.$$

If there are two functions u_1 and u_2 defined in $D+S$ such that $u_1/S=u_2/S$ and $Lu_1>Lu_2$, then $u_1\leqq u_2$ everywhere in D. It is also proved that if the inner diameter of D is sufficiently small, there exist functions B_1 and B_2 such that (*) is fulfilled. (In equation (2), p. 335, the term $a_{22}\,\partial^2 u/\partial x_2^2$ is omitted.) *J. Cronin* (New York, N.Y.).

Duff, G. F. D. A mixed problem for normal hyperbolic linear partial differential equations of second order. Canad. J. Math. **9** (1957), 141–160.

The mixed problem in question arises in specifying Cauchy data on a space-like initial surface, and an outer derivative on an intersecting time-like boundary surface. There are certain restrictions on the direction of the outer derivative. There are also, at least for smooth solutions, certain compatibility conditions relating the initial and boundary data at the intersection of the initial and boundary surfaces. The analytic case is treated first, by a method of majorants. The extension to the non-local non-analytic case is archieved by methods akin to those of Krzyzanski and Schauder [Studia Math. **6** (1936), 162–189]. *R. McKelvey* (Boulder, Colo.).

Diaz, J. B.; and Ludford, G. S. S. On the integration methods of Bergman and Le Roux. Quart. Appl. Math. **14** (1957), 428–432.

Soit l'équation linéaire hyperbolique

$$(1)\qquad Lu=u_{xy}+a(x, y)u_x+b(x, y)u_y+c(x, y)u=0.$$

Des représentations des solutions de (1) sont fournies soit par les formules de Bergman [Proc. Symposia Appl. Math., v. 1, Amer. Math. Soc., 1949, pp. 19–40; MR **11**, 222] soit par les formules de Le Roux [Ann. Sci. Ecole Norm. Sup. (3) **12** (1895), 227–316].

Les A. ont déjà explicité [Quart. Appl. Math. **12** (1955), 422–427; MR **16**, 369] une correspondance entre ces deux représentations. Les A. donnent ici une nouvelle correspondance entre ces deux représentations.

J. L. Lions (Nancy).

Pogorzelski, W. Sur le problème de Fourier généralisé. Ann. Polon. Math. **3** (1956), 126–141.

Consider the equation

$$(*)\qquad \sum_{\nu=1}^{n}\frac{\partial^2 u}{\partial x_\nu^2}-\frac{\partial u}{\partial t}=F(A, t, u),$$

where $(x_1, \cdots, x_n) \in \Omega$ a bounded domain in n-space with boundary S of Liapounoff type, A is a point of Ω, and the following boundary and initial conditions are imposed: (i) for $0<t\leq T$, $(du/dn_P)_t=\Phi(P, t, u_P)$ for all $P \in S$ and (ii) $\lim_{t\to 0} u(A, t)=0$ for all $A \in \Omega$. Suppose that $F(A, t, u)$ is continuous in t and satisfies a Hölder condition in (A, u); and that $\Phi(P, t, u)$ is continuous in (P, t, u). Assume also that several inequalities relating the upper bounds of $|F|$ and $|\Phi|$ (inequalities too complicated to be described here) are satisfied. By using the Schauder fixed-point theorem, the author proves that (*) has at least one solution u satisfying (i) and (ii). *J. Cronin.*

Sestini, Giorgio. Sulla risoluzione di un notevole gruppo di problemi retti da equazioni di tipo parabolico. Riv. Mat. Univ. Parma **6** (1955), 349–352.

The problem is: to determine the function $U=U(x, t)$ satisfying the following system of equations

$$\frac{\partial^2 U}{\partial x^2}-\frac{\partial U}{\partial t}=a(x, t)\ \frac{\partial U}{\partial x}+b(x, t)U+c(x, t) \qquad (0<x<l,\ 0<t<T)$$

$$U(x, 0)=f(x) \quad (0<x<l)$$

$$\alpha_1 U_x+\beta_1 U=g_1(t) \quad (x=0,\ 0<t<T)$$

$$\alpha_2 U_x+\beta_2 U=g_2(t) \quad (x=l,\ 0<t<T)$$

where α_i, β_i are given constants, $f(x)$, $g_i(t)$ are given continuous functions which satisfy a Lipschitz condition of order $\gamma\leq 1$, $a(x, t)$, $b(x, t)$, $c(x, t)$ are given continuous functions which satisfy a Lipschitz condition of order $\gamma\leq 1$ with respect to at least one of the variables in R_T, $0\leq x\leq l$, $0\leq t\leq T$.

The system of equations is equivalent to the integro-differential equation

$$U(x, t)=Z(x, t)-\frac{1}{2\sqrt{\pi}}\iint_{R_t}\left[a(\xi, \tau)\frac{\partial U}{\partial \xi}+b(\xi, \tau)U\right]G(\xi, \tau; x, t)\,d\xi d\tau,$$

where Z is the solution of the problem when $a=b\equiv 0$ and G is the corresponding Green's function. The conditions imposed are sufficient to ensure the uniqueness of the solution and the uniform convergence of the obvious procedure for solving the integro-differential equation by successive approximation. *E. T. Copson* (St. Andrews).

★ **Hopf, Eberhard. Repeated branching through loss of stability, an example.** Proceedings of the conference on differential equations (dedicated to A. Weinstein), pp. 49–56. University of Maryland Book Store, College Park, Md., 1956.

In a previous paper [Comm. Appl. Math. **1** (1948), 303–322; MR **10**, 716] the author constructed a model of turbulence, i.e., a system of nonlinear integrodifferential equations the solutions of which exhibit the same behavior that the solutions of certain hydrodynamical problems are surmised to have. In the present paper, a considerably simpler model is constructed:

$$(*) \qquad w_t'=\mathfrak{T}(w)+\mathfrak{L}(w)+\mu w_x'',$$

where $w(x, t)$ is an unknown complex valued function of time t and the angle variable x mod 2π, and $\mu>0$ is a parameter which corresponds to viscosity in the hydrodynamical equations. The operators $\mathfrak{T}(w)$ and $\mathfrak{L}(w)$ are defined as:

$$\mathfrak{L}(w)=\frac{1}{2\pi}\int_0^{2\pi} w(x+y)F^*(y)\,dy,$$

$$\mathfrak{T}(w)=-\frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} w(y)w(y')w^*(y+y'-x)\,dy dy',$$

where $F(y)$ is a given integrable periodic function of y and * indicates complex conjugate.

The same properties as for the previous system are obtained. There is an infinite set of critical values $\mu_1>\mu_2>\mu_3>\cdots\to 0$ such that if $\mu\geq\mu_1$, there is a time-independent (the laminar) solution $u=U(x; \mu)$ of (*) toward which all other solutions with $\mu>\mu_1$ converge as $t\to\infty$. In the space of functions $w(x)$, periodic in x of period 2π, a solution $w(x, t)$ of (*) describes a trajectory. For $n\geq 1$ and for each μ such that $\mu_n>\mu\geq\mu_{n+1}$, there is in S an n-dimensional manifold $M_n(\mu)$ such that the trajectories described by the "majority" of the solutions of (*) for the given value of μ converge to $M_n(\mu)$ as $t\to\infty$. Also as μ decreases, manifold $M_n(\mu)$ branches continuously away from $M_{n-1}(\mu_n)$.

It is indicated that the type of statistical theory developed for the previous model can also be developed for this model. *J. Cronin* (New York, N.Y.).

Oleĭnik, O. A. On discontinuous solutions of non-linear differential equations. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 1098–1101. (Russian)

$\varphi(f, x, u)$ sei bestimmt als Funktion der x, u und nichtnegativen Werte t mit stetigen Ableitungen zweiter Ordnung. $\varphi_u'(t, x, u)$ sei beschränkt für alle Werte x und $t\geq 0$, sobald u in endlichen Intervallen variiert. φ_{uu}'' sei nichtnegativ. — Die beschränkte und integrable Funktion $u(t, x)$ heißt eine Lösung des Cauchyschen Problems der parteillen Differentialgleichung erster Ordnung

$$\frac{\partial u}{\partial t}+\frac{\partial \varphi(t, x, u)}{\partial x}=0$$

in der Halbebene $t\geq 0$ mit der Anfangsbedingung $u(0, x)=u_0(x)$, wenn (1) für jede stetig differenzierbare Funktion $f(t, x)$, die ausserhalb eines gewissen endlichen Gebietes verschwindet, die Gleichung

$$\iint_{t\geq 0}\left[\frac{\partial f}{\partial t}u(t, x)+\frac{\partial f}{\partial x}\varphi(t, x, u(t, x))\right]dx dt+\int_{-\infty}^{+\infty} f(0, x)u_0(x)\,dx=0$$

erfüllt ist (worin sich das erste Integral über die Halbebene $t\geq 0$ erstreckt) und wenn (2) in jedem endlichen Gebiet D eine monoton abnehmende Funktion $K(t)$ angegeben werden kann, die für $t>0$ bestimmt ist und die Eigenschaft hat, daß

$$\frac{u(t, x_1)-u(t, x_2)}{x_1-x_2}\leq K(t)$$

gilt für irgend zwei Punkte (t, x_1) und (t, x_2) aus dem Gebiet D und $t>0$. Weiterhin wird nun die Existenz und Eindeutigkeit solcher Lösungen des Cauchyschen Problems bewiesen. Zum Schluß werden die erzielten Ergebnisse mit dem Cauchyschen Problem der parabolischen Differentialgleichung

$$\varepsilon^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}+\frac{\partial \varphi(t, x, u)}{\partial x}$$

in Verbindung gebracht. *M. Pinl* (Köln).

Bicadze, A. V. On the problem of equations of mixed type in many dimensional regions. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 901–902. (Russian)

Let D be the region in the space of $(x_1, \cdots, x_n, t)$ bounded by the hemisphere σ: $r^2+t^2=1$, $t\geqq 0$, and by the characteristic cones k_1: $t=-1+r$, $-\frac{1}{2}\leqq t\leqq 0$ and k_2: $t+r=0$, $-\frac{1}{2}\leqq t<0$, where $r^2=x_1^2+\cdots+x_n^2$. The problem is to determine a function $u(x_1, \cdots, x_n, t)$ such that: (a) u is a solution of the equation $\Delta u+\operatorname{sgn} t\cdot u_{tt}=0$ in D for $t\neq 0$; (b) u is continuous in the closure $\bar{D}$ of D; (c) all first-order partial derivatives of u are continuous in $\bar{D}$ everywhere except perhaps at the origin and on the manifolds $r=1$, $t=0$ and $r=\frac{1}{2}$, $t=-\frac{1}{2}$; (d) $u=\varphi$ on σ and $u=\psi$ on k_1. The author obtains a solution for $n=3$ depending only on r and t by means of the theory of harmonic functions in the (r, t)-plane. This problem is a step toward generalization of the author's previous study of the case $n=1$ [Trudy Mat. Inst. Steklov. **41** (1953); MR **16**, 43].

R. N. Goss (San Diego, Calif.).

Eĭdel'man, S. D.; and Lipko, B. Ya. On a theorem of Liouville. Mat. Sb. N.S. **40(82)** (1956), 273–280. (Russian)

Soit $U(x, t)$ un vecteur ($x\in R^n$, t=temps) solution pour $t\leqq T$ du système parabolique au sens de Petrowsky:

$$(1)\qquad \frac{\partial U}{\partial t}=P\Big(t, x, \frac{1}{2\pi i}\frac{\partial}{\partial x}\Big)U,$$

P matrice différentielle en x d'orche $2b$, à coefficients dépendant de x et t. On suppose que

$$|U(x, t)|\leqq\psi(t)\exp(k|x|),$$

k constante positive, $\psi(t)$ étant assujettie à une condition de croissance au voisinage de $-\infty$ (condition (24) des A.). Si le système (1) est assujetti à une condition (L) de majoration de la matrice de Green et de ses dérivées spatiales alors $U\equiv 0$.

La condition (L) est vérifiée par divers systèmes étudiés par Eĭdel'man [Mat. Sb. N.S. **33(75)** (1953), 359–382; **38(80)** (1956), 51–92; MR **15**, 712; **17**, 857] et par Gel'fand et Šilov [Uspehi Mat. Nauk (N.S.) **8** (1953), no. 6(58), 3–54; MR **15**, 867].

J. L. Lions (Lawrence, Kan.).

Šilov, G. E. On a Phragmen-Lindelöf type theorem for a system of linear partial differential equations. Trudy Moskov. Mat. Obšč. **5** (1956), 353–366. (Russian)

Considérons le système d'équations linéaires aux dérivées partielles:

$$(1)\qquad \frac{\partial u(x, t)}{\partial t}=P\Big(\frac{1}{2\pi i}\frac{\partial}{\partial x}\Big)u(x, t),$$

où $x=\{x_1, x_2, \cdots, x_N\}$ est un point de l'espace R_N, $u(x, t)=\{u_1(x, t), \cdots, u_m(x, t)\}$ une fonction (complexe) inconnue, à valeurs dans un espace vectoriel, P une matrice (m lignes, m colonnes) dont les éléments sont des polynomes par rapport aux opérateurs $(2\pi i)^{-1}\partial/\partial x_j$ avec des coefficients constants. L'auteur démontre le théorème suivant: Si pour tout $s=(s_1, s_2, \cdots, s_N)$ les racines caractéristiques de $P(s)$ sont réelles, toute solution du système (1) satisfaisant aux conditions: $|u(x, t)|\leqq Ce^{\alpha|t|+A|x|^\gamma}$ $\gamma<1$; $|u(x, 0)|\leqq C(1+|x|)^q$ (q entier non négatif) est de la forme $u(x, t)=\sum_{k=0}^{r} u^{[k]}(x)t^k$ ($r\leqq q+[\frac{1}{2}N]+m+1$), où les fonctions (vecteurs) $u^{[k]}(x)$ ($0\leqq k\leqq r$) sont des intégrales des équations

$$P\Big(\frac{1}{2\pi i}\frac{\partial}{\partial x}\Big)u^{[r]}(x)=0,$$

$$P\Big(\frac{1}{2\pi i}\frac{\partial}{\partial x}\Big)u^{[k-1]}(x)=ku^{[k]}(x)\ (k\leqq r).$$

Dans le cas particulier simple ($N=1$, $m=1$)

$$\frac{\partial u(x, t)}{\partial t}=i\frac{\partial u(x, t)}{\partial x}$$

on obtient un résultat sur les fonctions analytiques qui résulte aussi du théorème classique de Phragmen-Lindelöf. Le résultat ainsi obtenu est assez près du meilleur résultat possible (où $e|x|$ remplace $A|x|^\gamma$, $\gamma<1$, dans l'hypothèse).

S. Mandelbrojt (Paris).

See also: Nedelcu, p. 646; Huber, p. 650; Davis, Scott, Springer and Resch, p. 653; Šilov, p. 660; Favard, p. 668; Collatz, p. 677; Klein, p. 689; Žautykov, p. 691; Stewart and Ormsbee, p. 692; Tatsumi, p. 694; Dombrovskiĭ, p. 695; Holt, p. 695; Vodička, p. 701; Agekyan, p. 705.

Difference Equations, Functional Equations

Parker, R. V. Stirling's numbers as polynomials. Tôhoku Math. J. (2) **8** (1956), 181–182.

In this brief note the author shows how to construct an auxiliary table from which Stirling's numbers of the first and second kinds can be computed.

T. Fort.

See also: Peyerimhoff, p. 651; Ramakrishnan et Srinivasan, p. 680; Burger, p. 707; Bass, p. 709.

Integral and Integrodifferential Equations

Mainra, V. P. Certain integral equations and self-reciprocal functions. Ganita **6** (1955), 55–73 (1956).

The author considers the integral transform whose kernel is the resultant of a number of Hankel kernels. When f is self-reciprocal under one of these transforms the author obtains integral equations satisfied by $\sum f(nx)$ and $\sum(-1)^n f((2n-1)x)$. He also gives converse theorems and considers some other series of the same nature.

R. P. Boas, Jr. (Evanston, Ill.).

Parasyuk, O. S. On "paired" integral equations in the class of generalized functions. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 957–958. (Russian)

Remarques sur l'extension de la méthode de Wiener-Hopf aux distributions.

{Le problème ne semble pas posé de façon assez précise: si g_1 est une distribution sur R, „g_1 est nulle pour $x\geqq 0$" n'a pas de sens en général. Moyennant des hypothèses supplémentaires la méthode est valable.}

J. L. Lions (Lawrence, Kan.).

Amerio, Luigi. Sulle equazioni integrali con nucleo simmetrizzabile. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. **90** (1956), 141–155.

The kernel $K(x, y)$ in the square $0\leqq x\leqq 1$, $0\leqq y\leqq 1$, is said to be symmetrizable on the left by the symmetric kernel $H(x, y)$ if the function $g(x, y)=\int_0^1 H(x, z)K(z, y)dz$ is symmetric in x and y. K is assumed to be bounded, and H summable and either may be real or complex. The author

considers the integral equation

$$\varphi(x)=\lambda\int_0^1 K(x,y)\varphi(y)dy+f(x)$$

in the class of summable functions φ, f. If λ is not an eigenvalue of the integral equation, it was shown by S. Albertoni [same Rend. 87 (1954), 400–432, pp. 411–413; MR 17, 474] that the resolvent kernel $\Gamma(x, y)$ is also symmetrizable on the left by H. In the present paper, it is shown that if λ is an eigen value, this is no longer the case, but that in general

$$\int_0^1 H(x,z)\Gamma(z,y)dz=$$

$$\int_0^1 H(y,z)\Gamma(z,x)dz-\sum\{\psi_i(x)w_i(y)-\psi_i(y)w_i(x)\},$$

where the ψ_i are the eigenfunctions corresponding to the transposed kernel $K(y, x)$, and the w_j are certain functions (not in general identically zero) orthogonal to all the ψ_i.

J. W. Green (Stanford, Calif.).

★**Kohn, J. J.; and Spencer, D. C. Complex Neumann problems on Kähler manifolds.** Princeton University, Princeton, N. J., 1956. 115 pp. (mimeographed)

The authors make contributions to, and generalize from scalars to forms, a procedure of G. Giraud (and S. G. Mihlin) for inverting an operator

$$f(x)=I\rho=a(x)\rho(x)+\int_M G(x,y)\rho(y)dV_y$$

in which as in certain instances of the Neumann boundary problem the kernel $G(x, y)$ is not Lebesgue integrable in y in the neighborhood of $y=x$ but behaves as follows. If M is an m-dimensional Riemannian space and $r(x, y)$ its geodesic distance, then in all tangential directions the $\lim_{y\to x} r(x,y)^m G(x,y)$ exists and has average 0 over the sphere of directions. The problem is regularizable if a second such kernel can be found so that the equation $KI\rho=Kf$ has the same solutions as the original one but it of manageable Fredholm type itself.

Next, for complex Kaehler structure the authors combine the two differentiations ∂, $\bar\partial$ into one by introducing

$$d\tau=\tfrac{1}{2}(\tau-i)\partial+\tfrac{1}{2}(\tau+i)\bar\partial$$

and in order to include $\partial+\bar\partial$ they admit points τ of the entire complex Gauss sphere. There is a Neumann problem associable with each τ, and it can be regularized for "all" τ except $\tau=ri$ where r is a rational number in the closed interval $-1\leqq r\leqq 1$. The leading exception, is $\tau=i$. Its Neumann problem pertains to the construction of holomorphic functions from boundary values, and for more than one complex variable this is a significant problem indeed.

S. Bochner (Princeton, N.J.).

See also: Sestini, p. 656; Tautz, p. 661; Toraldo di Francia, p. 698; Kaprielian, p. 699; Bellman and Kalaba, p. 705.

Calculus of Variations

Miller, P. H., Jr. Hamilton's principle as a computational device. Amer. J. Phys. 25 (1957), 30–32.

An approximate method for finding the period of a periodic solution of a dynamical system. In the examples considered the systems are of integrable type and the period is obtained approximately as a function of an "amplitude", as in the case of a simple pendulum whose equations admit of exact integration with the help of elliptic functions. The method is to assume that an approximate solution may be expressed as a linear combination of relatively simple functions, such as trigonometric functions. The coefficients in this linear combination are then chosen so as to extremize Hamilton's integral. This can give rise to many false solutions, the elimination of which may presumably be accomplished by a proper "understanding of the physics".

D. C. Lewis, Jr. (Baltimore, Md.).

See also: Gel'fer, p. 648.

TOPOLOGICAL ALGEBRAIC STRUCTURES

Topological Groups

Wallace, Alexander Doniphan. Ideals in compact connected semigroups. Nederl. Akad. Wetensch. Proc. Ser. A. 59=Indag. Math. 18 (1956), 535–539.

Some general theorems (mostly of a technical nature) on ideals of (Hausdorff) topological semigroups S are given together with some applications. The applications consist of a theorem related to a paper of Mostert and Shields [Proc. Amer. Math. Soc. 7 (1956), 942–947; MR 18, 317] and the following. If S is an indecomposable continuum (=compact connected Hausdorff space), then no maximal subgroup of S cuts S. If S is compact, and every proper retract of S has the fixed point property, then S is a group or every element of the minimal ideal of S is an idempotent. If S is an n-sphere, and if some subset of a maximal subgroup cuts S, then this maximal subgroup is a Lie group and irreducibly cuts S. {Much of the terminology and notation used is not defined in the present paper. The author refers the reader to his address [Bull. Amer. Math. Soc. 61 (1955), 95–112; MR 16, 796]. Also, reference [8] should be changed to [1].}

M. Henriksen.

Schieferdecker, Eberhard. Einbettungssätze für topologische Halbgruppen. Math. Ann. 131 (1956), 372–384.

This paper generalizes and provides new proofs for results of an earlier paper of the author [Math. Z. 62 (1955), 443–486; MR 17, 384; see that review for notations, definitions and references] from the metric case to that of a possibly nonseparated uniform structure with various invariance properties: extendability of uniform structures of a semigroup H to $Q_l(H; m)$ applied to topological embeddability of cancellation semigroups H in $Q_l(H; H)$ as a topological group, and of Ore semigroups H in $Q_l(H; H)$.

G. K. Kalisch (Minneapolis, Minn.).

See also: Tominaga, p. 636; Masuda, p. 643; Tashiro, p. 658; Dedò, p. 674; Kloss, p. 680; Rivkind, p. 680.

Lie Groups and Algebras

Tashiro, Yoshihiro. On extensions of Lie group, transformation group and fibre bundle. Math. J. Okayama Univ. 6 (1956), 99–107.

In this paper Ehresmann's theory of jets and extensions

(prolongements) [C. R. Acad. Sci. Paris **233** (1951), 598–600, 777–779, 1081–1083; **234** (1952), 1028–1030, 1424–1425; MR **13**, 386, 584, 780, 870] is applied to Lie groups and fibre bundles. Let X be a differentiable manifold of class C^s, and let R^k be the k-dimensional Euclidean space. We denote by $T_k^r(X, x)$ the space of r-jets of local maps $f: R^k, 0 \to X, x$. The union $T_k^r(X) = \mathbf{U}_{x \in X} T_k^r(X, x)$ can be given the structure of a differentiable manifold of class C^{s-r} $(s \geqq r)$ and is called the k^r-extension of X. The author shows that the k^r-extension of a Lie group is, in a natural way, a Lie group, and he gives an explicit formula for the group structure in the case $k=r=1$. Moreover if B is a fibre bundle of class C^s over X with group G and fibre Y, then $T_k^r(B)$ is a fibre bundle of class C^{s-r} over $T_k^r(X)$ with group $T_k^r(G)$ and fibre $T_k^r(Y)$. Finally it is shown that the operations of forming (a) the k^r-extension, and (b) the associated principal bundle, commute.

M. F. Atiyah (Cambridge, England).

Dixmier, J.; and Lister, W. G. Derivations of nilpotent Lie algebras. Proc. Amer. Math. Soc. **8** (1957), 155–158.

N. Jacobson has proved [same Proc. **6** (1955), 281–283; MR **16**, 897] that, over a field of characteristic 0, a Lie algebra with a non-singular derivation is nilpotent. He pointed out that the question of the validity of the converse was open. Specifically: If L is a nilpotent Lie algebra, does L have a derivation without non-zero constants? The present paper provides a negative answer to this question with the explicit construction of an algebra L of dimension 8. L is not the derived algebra of any Lie algebra and L is characteristically nilpotent, in the following sense. If N is an algebra with derivation algebra D, let $N^{(1)}=ND=[\sum x_i D_i | x_i \in N, D_i \in D]$, $N^{(k+1)}=N^{(k)}D$. N is called characteristically nilpotent if, for some k, $N^{(k)}=0$. The authors ask whether there is an intrinsic characterization for such algebras and whether can there be found a general method for constructing them all.

R. L. San Soucie (Eugene, Ore.).

Gurevič, G. B. Some properties of standard Lie nulalgebras. Trudy Sem. Vektor Tenzor. Anal. **10** (1956), 89–104. (Russian)

The author proves that any standard matric nulalgebra [Gurevič, Mat. Sb. N.S. **35(77)** (1954), 437–460; MR **17**, 509] may be represented as the intersection of a special system of the so-called full nulalgebras. Further, an enumeration of the types of standard matric nulalgebras is given. *W. T. van Est* (Leiden).

Lyndon, R. C. A theorem of Friedrichs. Michigan Math. J. **3** (1955), 27–29.

Let Φ the free associative ring over a field of characteristic zero in the noncommuting indeterminates $x_1, x_2, \cdots$ and Λ the K-submodule of Φ generated by $x_1, x_2, \cdots$ under the operation of forming Lie-products $[G, H]=GH-HG$. The elements of Λ (Lie-element of Φ) can be characterized according to a theorem of Friedrichs in the following way: An element $F(x_1, x_2, \cdots)$ of Φ belongs to Λ if and only if the relations $x_i' x_j' = x_j'' x_i'$ imply

$$F(x_1'+x_1'', x_2''+x_2'', \cdots) = F(x_1', x_2', \cdots)+F(x_1'', x_2'', \cdots).$$

Using an operator introduced by Dynkin, Specht and Wever the author obtains a new simple proof of Friedrichs' theorem. *C. Loewner* (Palo Alto, Calif.).

See also: Seligman, p. 638; Wallace, p. 658; Murakami, p. 663; Weil, p. 673.

Topological Vector Spaces

Deprit, André. Sous-espaces vectoriels d'un espace localement convexe séparé. Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 1012–1017.

In connection with his investigations of linear transformations [J. Fac. Sci. Univ. Tokyo. Sect. I. **7**, 129–192 (1954), 305–332 (1956); MR **16**, 992; **17**, 1044] Hukuhara established a result concerning the meshing of two chains of subspaces. The author puts this result in the setting of locally convex spaces, making one chain an ascending one of finite-dimensional subspaces, the other a descending one of closed cofinite subspaces. *I. Kaplansky.*

Audin, Maurice. Sur les transformations linéaires des espaces vectoriels qui satisfont à une condition de Fredholm. C. R. Acad. Sci. Paris **244** (1957), 711–713.

Let E be a real or complex vector space and G a subspace of the algebraic dual E^* of E. The intersection of all G-hyperplanes (which are defined by equations $\langle x, y\rangle=0$, $y \in G$) is denoted by E_0. If $E_0 \neq 0$ the author extends some results which are evident in the case $E_0=0$, for instance: if N is a linear subspace of E of finite codimension, then N is the intersection of a finite number of G-hyperplanes if and only if N is closed in a linear topology on E of which the dual is isomorphic with G. Let $\mathfrak{A}$ and $\mathfrak{B}$ be the family of all the linear transformations of E in E and of G in G respectively. If $T \in \mathfrak{A}$ and $T' \in \mathfrak{B}$, then T' is called the G adjoint of T if $\langle Tx, y\rangle=\langle x, T'y\rangle$ for all $x \in E$, $y \in G$. A pair (T, U) with $T \in \mathfrak{A}$ and $U \in \mathfrak{B}$ is said to be normal if the equation $Tx=b$ has at least one solution in E if and only if $\langle b, y\rangle=0$ for all y with $Uy=0$. (Fredholm condition): (T, U) is binormal if the same condition is satisfied changing the roles of T and U. T is called G-normal (G binormal) if (T, T') is normal (binormal). If $\mu(T)$ is the dimension of $T^{-1}(0)$ of T and $\mu^*(T)$ the codimension of $T(E)$ and if T has G-adjoint with $\mu(T')$ and $\mu^*(T)$ finite, then T is normal if $\mu(T')=\mu^*(T)$. If E is a locally convex linear topological Hausdorff space then a continuous linear transformation T of E into E is binormal if $T(E)$ is closed. These are some of the results the author announces in this paper; they will be applied in a subsequent paper, to extend results of Nikol'skiĭ [Izv. Akad. Nauk SSSR. Ser. Mat. **7** (1943), 147–166; MR **5**, 187] and Gohberg [Dokl. Akad. Nauk SSSR (N.S.) **76** (1951), 477–480; MR **13**, 46]. *W. A. J. Luxemburg* (Toronto, Ont.).

Nef, Walter. Über die Fortsetzung monotoner Linearformen. Math. Z. **66** (1956), 129–142.

In a previous paper [Monatsh. Math. **60** (1956), 190–197; MR **18**, 321] the author discussed conditions in order that a linear form defined on a subspace L of a partially ordered linear space M should have a positive extension to the entire space. This article discusses the values which the possible extensions can take for a given y of the space. Let M be partially ordered with the cone of positive and zero elements P, F_0 a positive linear form on L, y a point not in L. Let Q be the set of all cones P^* which contain P, induce an ordering of L in which F_0 is positive, and are such that for some $x_0 \in L$ there is a convex set K such that $x_0+K \subset P^*$ and the sets λK, for all real λ include all M. Let $\bar{p}(y)=\sup \inf F_0(x)$, the infimum being over all x in L greater than y in P^* ordering, the supremum over all

P^* in Q, and $\underline{p}(y)$ is defined similarly with reversed inequalities. Then for every η in $(\underline{p}(y), \bar{p}(y))$ there is a P-positive extension F of F_0 such that $F(y)=\eta$. The set for which $\underline{p}(y)=\bar{p}(y)$ is a subspace $L^\times$ of M; the mapping L to $L^\times$ is a closure operation. *J. L. B. Cooper.*

Bauer, Heinz. Sur le prolongement des formes linéaires positives dans un espace vectoriel ordonné. C. R. Acad. Sci. Paris **244** (1957), 289–292.

A theorem of Krein on extensions of linear functionals [see Bourbaki, Espaces vectoriels topologiques, Actualités Sci. Ind., no. 1189, Hermann, Paris, 1953, p. 75, Prop. 6; MR **14**, 880] is generalised as follows: Let P be the cone of positive elements in an ordered vector space E, and V the set of elements for which a seminorm p satisfies $p(x)<1$, M a subspace, and f a linear form on M. Then, in order that there exist a positive linear form F which is an extension of f to the whole space, and which satisfies $|F(x)|\leq p(x)$ everywhere, it is necessary and sufficient that, for every x in $M\cap(V+P)$, $f(x)>-1$.

If E is a locally convex topological space, then for f to be capable of extension to a positive continuous function on E, it is necessary and sufficient that there be at least one neighbourhood of 0 in E, say V, for which

$$f(M\cap(V+P))$$

is bounded below. This theorem generalises a theorem of W. Nef in the paper referred to in the preceding review. As an application, it is shown that for a continuous positive form to exist which has the value 1 at a point x, it is necessary and sufficient that $-x$ is not in the closure of P: no continuous positive linear functional exists if P is everywhere dense in E. *J. L. B. Cooper* (Cardiff).

Amemiya, Ichiro. On the unconditional convergence in semi-ordered linear spaces. J. Fac. Sci. Hokkaido Univ. Ser. I. **13** (1956), 54–59.

The type of convergence mentioned in the title is order convergence [see H. Nakano, Modern spectral theory, Maruzen, Tokyo, 1950; MR **12**, 419]. Necessary and sufficient conditions are given for the equivalence of unconditional and absolute convergence (with examples of classes of spaces satisfying them, or not satisfying them, respectively), as well as sufficient conditions for the equivalence of the two types of convergence (again with various examples). The conditions are too lengthy to be stated here. *G. K. Kalisch* (Minneapolis, Minn.).

Roumieu, Charles. Sur le problème aux valeurs initiales pour une équation de convolution homogène à une variable. C. R. Acad. Sci. Paris **244** (1957), 430–432.

p étant réel, $m\geqq 0$, désignons par $A^+(p, m)$ la classe des distributions φ de segment-support $(0, l)$, telles que $\lim (iw)^{-p}\Phi(w)=M$ avec $0<|M|<\infty$, uniformément lorsque $v+m\ \mathrm{Log}\ (1+|u|))$ tend vers $-\infty(w=u+iv)$, Φ étant la transformée de Fourier de φ [$\Phi(w)=\varphi\cdot e^{-iwt}$ (produit scalaire)]. Ou désignera aussi par $A^-(p, m)$ la classe pour laquelle $\lim(iw)^{-p}\, e^{liw}\Phi(w)=M$, avec $v+m\ \mathrm{Log}(1+|u|)\to -\infty$, uniformément lorsque $v-m\ \mathrm{Log}(1+|u|)\to+\infty$. Si $\varphi\in A^+(p, m)\cap A^-(p', m')$, y étant une distribution telle que $\varphi * y=0$ sur $(0, \varepsilon)$, il existe une et une seule distribution x telle que $x=y$ sur $(-l, \varepsilon)$ avec $\varphi * x=0$ sur la droite. *S. Mandelbrojt.*

Tanabe, Hiroki. On spectral theory of completely continuous and some other closed operators. Sci. Papers Coll. Gen. Ed. Univ. Tokyo **6** (1956), 13–21.

Let T be a closed linear operator in a B-space X with dense domain and let σ be a bounded spectral set of T. If, for some positive integer n and some λ in σ, the range of $(\lambda I-T)^n$ is closed, then the equation $y=(\lambda I-T)^n x$ has a solution x if and only if y is orthogonal to the null manifold of $(\bar{\lambda}I^*-T^*)^n$. Also $g=(\bar{\lambda}I^*-T^*)^n f$ has a solution f if and only if g is orthogonal to the null manifold of $(\lambda I-T)^n$. (For a linear functional x^* and a complex number λ the functional λx^* is defined by the equation $(\lambda x^*)x=\bar{\lambda}x^*(x)$.) If the spectrum $\sigma(T)$ of T is at most countable and has no cluster point other than 0, then $\lim_{n\to\infty}(|T^n x|)^{1/n}=0$ if and only if $E_\lambda x=0$ for every $\lambda\neq 0$. Here E_λ is the projection obtained by integrating $(2\pi i)^{-1}(\xi I-T)^{-1}$ around a small circle containing λ and no other point of $\sigma(T)$. A number of other related results are given. *N. Dunford* (New Haven, Conn.).

Šilov, G. E. Generalized functions and their applications in analysis. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 6(72), 217–226. (Russian)

L'A. énumère (en général sans démonstration) un certain nombre de développements et applications récents de la théorie des distributions de L. Schwartz: 1) travail de Gel'fand-Šapiro sur les distributions homogènes; 2) généralisation de la transformation de Fourier et applications aux équations d'évolution (Gel'fand, Šilov et de nombreux autres auteurs Soviétiques); 3) théorème de Malgrange sur l'existence de solutions élémentaires des opérateurs différentiels à coefficients constants; 4) inégalités de Hörmander et applications; 5) application des espaces nucléaires de Grothendieck à la décomposition spectrale d'opérateurs différentiels (Gel'fand-Kostyučenko). *J. L. Lions* (Lawrence, Kan.).

Lyubovin, V. D. On spectral decomposition of self-adjoint operators. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 4(70), 139–142. (Russian)

Kantorovich, Vulich, and Pinsker [Functional analysis in partially ordered spaces, Gostehizdat, Moscow-Leningrad, 1950; Uspehi Mat. Nauk (N.S.) **6** (1951), no. 3(43), 31–98; MR **12**, 340; **13**, 361] established the spectral representation theorem for bounded self-adjoint operators as follows: If A is bounded and selfadjoint and if

$$A'=\{B|\ B \text{ bounded self-adjoint, } AB=BA\},$$

$$A''=\{C|\ C \text{ bounded self-adjoint, } CB=BC,\ B\in A'\},$$

then A'' is (in the language of K.V.P. for partially ordered vector spaces) a K-space with unit. Since $A\in A''$, the integral representtation for elements of K-spaces yields the spectral theorem. The author treats the case in which A is not bounded. For this, let R_λ be the resolvent of A, which exists if $\mathrm{Im}\ \lambda\neq 0$ and let $A_1=\frac{1}{2}(R_i+R_{-i})$, $A_2=\frac{1}{2}(R_i-R_{-i})=R_iR_{-i}$. Then A_1, $A_2\in A'$ and in fact A_2 is bounded, ($\|A_2\|\leq 1$) and self-adjoint. If A_2 has the spectral resolution $\{E_\lambda\}$, set $P_k=E_{2/k}-E_{2/(k+1)}$ ($k=1, 2, \cdots$), and let X_k be the component of A'' which contains P_k, and $H_k=P_kH$ (H=Hilbert space). The formula $\sum_1^\infty P_k=I$ is immediate. The fact that $A_k=AP_k\in A''$ is less obvious but established. If

$$X^*=\mathbf{S}_1^\infty X_k=\{B|\ B=\sup\{B_k|B_k\in X_k\}\},$$

let $\mathscr{A}$ be the self-adjoint elements in X^*, then $A''\subset\mathscr{A}$, $A\in\mathscr{A}$. $\mathscr{A}$ is easily seen to be K-space and the integral representation now follows. *B. R. Gelbaum.*

See also: Ghika, p. 638; Bauer, p. 645; Bharucha-Reid, p. 646; Mikusiński, p. 647; Parasyuk, p. 657; Donoghue, p. 661.

Banach Spaces, Banach Algebras

Tautz, G. L. Eigenwerte bei nichtsymmetrischen Operatoren. Arch. Math. **7** (1956), 310–316.

The author's principal result is as follows. Let $\mathfrak{K}$ be a compact set (in the uniform topology) of completely continuous linear operators in a Banach space; let $K_0 \in \mathfrak{K}$, and let λ_0 be a characteristic value of K_0. Then, given $\varepsilon>0$, there is a neighbourhood V of K_0 in $\mathfrak{K}$ such that every $K \in V$ has a characteristic value λ for which $|\lambda-\lambda_0|<\varepsilon$.

The author also shows that, in the particular case of Hilbert space, if $\mathfrak{K}$ is a compact set of real Hilbert-Schmidt operators and λ_0 is a real characteristic value of $K_0 \in \mathfrak{K}$ of multiplicity 1, then, for given $\varepsilon>0$, every K belonging to $\mathfrak{K}$ and sufficiently near K_0 has a real characteristic value λ for which $|\lambda-\lambda_0|<\varepsilon$.

{Reviewer's note. The author's definition of completely continuous operators is too wide; it would be satisfied by every bounded linear operator in (l^1). Partly as a result of this, some of his proofs are inadequate, but they can easily be modified; the results appear to be substantially correct.} *F. Smithies* (Cambridge, England).

Rudin, Walter. Les idéaux fermés dans un anneau de fonctions analytiques. C. R. Acad. Sci. Paris **244** (1957), 997–998.

Let A denote the Banach algebra of all complex-valued functions continuous on $|z|\leq 1$ and analytic for $|z|<1$ (with the usual sup norm). An explicit description of the closed ideals of A is given. From this description it follows that every closed ideal of A is generated by a single element. It also yields neat necessary and sufficient conditions for a closed ideal to be an intersection of maximal (respectively primary) ideals. In addition, the primary ideals of A are described explicitly. No proofs are given, but otherwise the note is self-contained. *M. Henriksen.*

See also: Bauer, p. 645; Fréchet, p. 646; Hufford, p. 654; Donoghue, p. 661.

Hilbert Space

Louhivaara, Ilppo Simo. Bemerkung zur Theorie der Nevanlinnaschen Räume. Ann. Acad. Sci. Fenn. Ser. A. I. no. **232** (1956), 7 pp.

[For notations and terminology see Nevanlinna, same Ann. nos. **108**, **113**, **115** (1952); **163** (1954); MR **14**, 287, 658; **15**, 717.] If U is closed subspace of the Nevanlinna space R and if $a \in R$, we have the following problem: Find $\{p|Q(a-p, u)=0, u \in U\}$. In Hilbert space (Nevanlinna space with a definite metric) the problem is of course trivial. U can be decomposed into linear subspaces U^+, U^0, U^-, where Q is respectively non-negative, 0, non-positive. If H is the majorizing Hermitian metric for Q, then there is a resolution of the identity $\{E_\lambda\}$ such that $Q(u, v)=\int_{-1}^{1} \lambda dH(E_\lambda u, v,)$, for $u, v \in U$. Furthermore $U^\pm=E^\pm U$, where $E^+=\int_{0<\lambda\leq 1} dE_\lambda$, $E^-=\int_{-1\leq\lambda<0} dE_\lambda$, and $U^0=\{u_0|Q(u_0, v)=0, v \in U\}$. The author gives the following sufficient condition for the existence of a solution to the problem: Let $U_x{}^0=E_x{}^0 U$, where $E_x{}^0=\int_{-x\leq\lambda<x} dE_\lambda$. Then there is an $x>0$ such that a is Q-orthogonal to $U_x{}^0$: $Q(a, u^0)=0$, all $u^0 \in U_x{}^0$. *B. R. Gelbaum.*

Donoghue, William F., Jr. Continuous function spaces isometric to Hilbert space. Proc. Amer. Math. Soc. **8** (1957), 1–2.

The note investigates the theorem of Banach [Théorie des opérations linéaires, Warszawa, 1932, p. 187] that any separable real Banach space is isometric to a subspace of $C(0, 1)$ of continuous functions on the unit interval for the case when the space is a real separable Hilbert space $\mathfrak{H}$. It finds that the subspace in question is pathological in the sense that if $N=\dim(\mathfrak{H})$ and $x_i(t)$ $(i=1, \cdots, k)$ are $k<N$ linearly independent function in the map of $\mathfrak{H}$ then the curve $x(t)=(x_i(t))$ is a space filling curve in the space R_k of dimension k. A similar result holds for any (separable) real Banach space with strictly convex dual.
T. H. Hildebrandt (Ann Arbor, Mich.).

Inoue, Sakuji. Simplification of the canonical spectral representation of a normal operator in Hilbert space and its applications. Mem. Fac. Ed. Kumamoto Univ. **3** (1955), suppl. no. 1, 50 pp.

Certain normal operators N in Hilbert space have their spectra in a curve and their resolutions of the identity may be written in the form $K(Z)=K(\lambda e^{i\mu})=E(\mu)F(\lambda)$, where E and F are the resolutions of the identity for the unitary operator $\mathfrak{U}$ and the positive self adjoint operator, H, respectively, in the polar representation $N=\mathfrak{U}H$ of N. For such operators a classification of spectral points and the problem of unitary equivalence are discussed.
N. Dunford (New Haven, Conn.).

See also: Moppert, p. 646; Lyubovin, p. 660; Tautz, p. 661; Krasnosel'skiĭ, p. 676.

TOPOLOGY

General Topology

Mycielski, Jan. On the decompositions of Euclidean spaces. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 417–418.

Three theorems concerning decompositions of E^3 are stated; the proofs are to appear later. The first theorem states that there exist a free group G of rank $\leq 2^{\aleph_0}$ of orientation preserving isometries of E^3 such that no isometry in G different from the identity has a fixed point. The following decomposition theorem follows from this. Let M and N be sets of potency $\leq 2^{\aleph_0}$. Suppose $\{P_\mu : \mu \in M\}$ and $\{Q_\mu : \mu \in M\}$ are collections of subsets of N distinct from 0 and N. Then there exists a decomposition of E^3 into disjoint sets $\{A_\nu : \nu \in N\}$ such that for each $\mu \in M$ the sets $\sum_{\nu \in P_\mu} A_\nu$ and $\sum_{\nu \in Q_\mu} A_\nu$ are congruent, with the congruence realized by an orientation preserving isometry. *E. E. Floyd.*

Mycielski, Jan. On the congruence of sets. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 419–421.

This note is a summary of several results concerning congruence in manifolds; proofs are to be published later. (1) Let M be an analytic manifold and G a free group of rank 2 of analytic homeomorphisms of M onto itself. Then there exists a countable set $E \subset M$, such that for each finite set $F \subset E$ there exists a homeomorphism $h_F \in G$ with $h_F(E)=E-F$. The existence of such a free group of rotations of S^2 is well-known; the author states that the existence of such groups of isometries of the elliptic and hyperbolic planes has been established by T. Dekker. (2) A free group G of rank $2^{\aleph_0}$ contains a set E

such that for any sets A, B which are at most denumerable there exists a $\phi \in G$ with $\phi E=(E-A)+B$. (3) There exists on the sphere S a set E such that $E\simeq(E-A)+B$ for any sets A, BCS which are at most denumerable.

E. E. Floyd (Charlottesville, Va.).

Banaschewski, Bernhard. Spaces of dimension zero. Canad. J. Math. **9** (1957), 38–46.

There is explored the connection between zero-dimensional, perfect, locally compact, metrizable spaces and the p-adic "Baire" spaces. It is also shown that there are only two (modulo uniform equivalence) perfect evenly-locally compact non-archimedean spaces: the 2-adic integers and the 2-adic numbers. *R. Arens.*

Grabar', M. I. On time transformations in dynamical systems. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 250–252. (Russian)

Let $\{S_t\}$ be a dynamical system (a one-parameter group of homeomorphisms) on a compact space R. The author considers a time transformation of the type

$$\tau=\phi(x, t)=\int_0^t F(S_\theta x)d\theta,$$

where $F(x)$ is a continuous positive function on R. He thus obtains a second dynamical system $\{\tilde{S}_t\}$ by putting $\tilde{S}_t x=S_t x$. Then if there exists a normalised, invariant and indecomposable measure m for the system $\{S_t\}$ and if $\int_0^t F(x)dm=1$, there exists a similar measure $\tilde{m}$ for the new system. One is now interested in whether the two systems are unitarily isomorphic. The theorem is stated and proved that they are isomorphic if there exists a function $\Phi(x)$ for which $\Phi(S_t x)=\Phi(x)+\int_0^t (F(S_\theta x)-1)d\theta$ almost everywhere. Indeed $T\tilde{S}_t=S_t T$ if $T(x)=S_{\phi(x)}x$. This theorem is followed by the statements of three corollaries which tend to drop the explicit mention of the function $\Phi(x)$. *G. Hufford* (Stanford, Calif.).

McCandless, Byron H. Test spaces for dimension n. Proc. Amer. Math. Soc. **7** (1956), 1126–1130.

Within the category of separable metric spaces, we write $X\tau Y$ [following Kuratowski, Topologie, vol. II, 2ème éd., Warszawa, 1952; MR **14**, 892] to mean that, if C is closed in X, then any $f: C\to Y$ may be extended to X. Consider the following five properties of a space $Y(n)$ $(n\geqq 1)$. 1^n: $X\tau Y(n)$ implies $\dim X\leqq n$; 2^n: $\dim X\leqq n$ implies $X\tau Y(n)$; 3^n: $\pi_i(Y(n))=0$ $(0\leqq i\leqq n-1)$; 4^n: $Y(n)$ is LC^{n-1} (locally connected in sense of homotopy in dimension m, or m-LC, for all $m\leqq n-1$); 5^n: $\pi_n(Y(n))$ is infinite cyclic if $n=1, \pi_n(Y(n))$ is free abelian on $k_n>0$ generators if $n>1$.

Kuratowski (loc. cit., p. 265) showed that 2^n is equivalent to (3^n and 4^n). Let $Y(n)$ be compact, n-dimensional and n-LC (and 1-simple if $n=1$). Then the author shows that if $Y(n)$ has properties 1^n and 2^n, it has property 5^n; and if $Y(n)$ has properties 3^n, 4^n, and 5^n, it has property 1^n.

$Y(n)$ is called a test space for dimension n if it has properties 1^n and 2^n. Thus the author concludes that, under the given hypotheses on $Y(n)$, $Y(n)$ is a test space for dimension n if and only if it has properties 3^n, 4^n, and 5^n. In fact, of course, this means that, under the given hypotheses on $Y(n)$, $Y(n)$ is a test space for dimension n if and only if it has the homotopy type of a union of n-spheres (or a circle if $n=1$). *P. J. Hilton.*

See also: Wolk, p. 635; Hilton, p. 639; Cecconi, p. 645; Neugebauer, p. 645; Fréchet, p. 646; Pogorzelski, p. 655; Wallace, p. 658; Amemiya, p. 660; Favard, p. 668.

Algebraic Topology

Moore, J. C. On a theorem of Borsuk. Fund. Math. **43** (1956), 195–201.

Borsuk [Fund. Math. **39** (1952), 25–37; MR **15**, 51] showed that if X is a compact space of dimension $\leqq k$ having positive kth Betti number then S^X, where S is the m-sphere, $m\geqq k$, has positive $(m-k)$th Betti number.

Let (X, A) be a finite dimensional compact pair and let $E(X, A)$ be the function space of maps $X, A\to S^m, y^m$. We use singular homology in $E(X, A)$ and Čech cohomology in (X, A). Theorem 1: If $H^q(X, A)=0$ for $q>k$, where $k\leqq m$, then $\pi_q(E(X, A))=0$, $q<m-k$ and $\pi_{m-k}(E(X, A))\cong H^k(X, A)$. The argument proceeds by induction on $m-k$, the case $m=k$ being the Hopf classification theorem. Corollary: $H_{m-k}(E(X, A))\cong H^k(X, A)$. To obtain further results in this direction, the author defines a homomorphism ϕ: $H_r(E(X, A))\to H^{m-r}(X, A)$ for any compact pair (X, A). Lemma 3. If $(X, A)=(S^k, y^k)$, $k<m$, and $n=\min(2(m-k)-1, m)$, then ϕ is an isomorphism for $0<r\leqq n$ and $H_r(E(S^k, y^k))=0$ for $n<r<2(m-k)$; this is proved from knowledge of the homology of iterated loop spaces of spheres. Theorem 2: If (X, A, B) is a triple of finite dimensional compact spaces, $H^r(A, B)=0$ for $r>p$, $H^r(X, A)=0$ for $r>q$, $m>\max(p, q)$, $s=2m-(p+q+1)$, $n=\min(s, m)$, then there is an exact sequence

$$H_s(E(X, A))\to\cdots\to H_r(E(X, A))\to H_r(E(X, B))\to$$
$$\to H_r(E(A, B))\to H_{r-1}(E(X, A))\to\cdots$$

and a diagram, commutative up to sign,

$$\begin{array}{ccccccc} H_n(E(X,A)) & \to\cdots\to & H_r(E(X,A)) & \to & H_r(E(X,B)) & \to & \\ \downarrow\phi & & \downarrow\phi & & \downarrow\phi & & \\ H^{m-n}(X,A) & \to\cdots\to & H^{m-r}(X,A) & \to & H^{m-r}(X,B) & \to & \\ & & \to H_r(E(A,B)) & \to & H_{r-1}(E(X,A)) & \to\cdots & \\ & & \downarrow\phi & & \downarrow\phi & & \\ & & H^{m-r}(A,B) & \to & H^{m-r+1}(X,A) & \to\cdots. & \end{array}$$

Using Lemma 3 on this diagram and the quoted facts if $(X, A)=(S^k, y^k)$, the author obtains an inductive proof of Theorem 3: If (X, A) is a k-dimensional compact pair, $k<m$, A is non-empty, $H^0(X, A)=0$, then $E(X, A)$ is connected, ϕ: $H_r(E(X, A))\cong H^{m-r}(X, A)$, $0<r<\min(2(m-k), m)$, and $H_r(E(X, A))=0$, $m\leqq r<2(m-k)$.

The author raises four problems concerned with further extensions of this result; in particular, the question is posed as to whether a spectral sequence exists relating the cohomology of X and the homology of $E(X)$.

{Reviewers note: The Corollary to Theorem 1, with A empty, has been proved by Mardešić independently. C. R. Acad. Sci. Paris **242** (1956), 2214–2217; MR **17**, 1118.} *P. J. Hilton* (Manchester).

James, I. M. On the suspension sequence. Ann. of Math. (2) **65** (1957), 74–107.

This is the fourth of a series of four papers; the results of the previous three papers [Ann. of Math. (2) **62** (1955), 170–197; **63** (1956), 191–247; 407–429, we refer to these papers as J1, J2, J3; MR **17**, 396, 1117; **18**, 58] are applied to obtain detailed information on homotopy groups of spheres and, in particular, the suspension

homomorphism.

The suspension sequence of S^n is the sequence

$$(1)\quad \cdots\to\pi_r(S^n)\xrightarrow{E}\pi_{r+1}(S^{n+1})\xrightarrow{i}\pi_{r+1}(S^{n+1};E_+,E_-)\xrightarrow{\Delta}\pi_{r-1}(S^n)\to\cdots,$$

where E_+, E_- are the Northern and Southern hemispheres of S^{n+1}. In J2 (using the notion of reduced product introduced in J1) the author defined a homomorphism

$$h:\pi_{r+1}(S^{n+1};E_+,E_-)\to\pi_{r+1}(S^{2n+1})$$

and proved in J3 that h is an isomorphism if n is odd and an isomorphism of 2-primary components if n is even (actually more was proved in J3 in the case n even). Thus, if n is odd, (1) gives rise to an exact sequence

$$\cdots\to\pi_r(S^n)\xrightarrow{E}\pi_{r+1}(S^{n+1})\xrightarrow{H}\pi_{r+1}(S^{2n+1})\xrightarrow{Q}\pi_{r-1}(S^n)\to\cdots,$$

where $H=hi$, $Q=\Delta h^{-1}$; h has the naturality property

$$\begin{array}{ccc}\pi_{r+1}(S^{m+1};E_+,E_-) & \to & \pi_{r+1}(S^{n+1};E_+,E_-)\\ \downarrow h & & \downarrow h\\ \pi_{r+1}(S^{2m+1}) & \xrightarrow{(\alpha*\alpha)_*} & \pi_{r+1}(S^{2n+1})\end{array}$$

where $\alpha\in\pi_m(S^n)$, so that the join $\alpha*\alpha\in\pi_{2m+1}(S^{2n+1})$; we recall that $\alpha*\alpha=E^{n+1}\alpha\circ E^{m+1}\alpha$. This naturally leads to a commutative diagram

$$\begin{array}{ccccccccc}\cdots\to\pi_r(S^m) & \xrightarrow{E} & \pi_{r+1}(S^{m+1}) & \xrightarrow{H} & \pi_{r+1}(S^{2m+1}) & \xrightarrow{Q} & \pi_{r-1}(S^m) & \to\cdots\\ \downarrow\alpha_* & & \downarrow(E\alpha)_* & & \downarrow(\alpha*\alpha)_* & & \downarrow\alpha_* & \\ \cdots\to\pi_r(S^n) & \xrightarrow{E} & \pi_{r+1}(S^{n+1}) & \xrightarrow{H} & \pi_{r+1}(S^{2n+1}) & \xrightarrow{Q} & \pi_{r-1}(S^n) & \to\cdots\end{array}$$

if m, n are odd. If m or n is even, a modification of the statement must be made. Let $\sigma:\pi_r(S^n)\to\pi_r(S^n)$ and $\rho:\pi_{r+1}(S^{n+1};E_+,E_-)\to\pi_{r+1}(S^{n+1};E_+,E_-)$ be induced by a map $S^n\to S^n$ of degree -1 and let $\sigma:\pi_r(S^{n+1})\to\pi_r(S^{n+1})$ be induced by a map $S^{n+1}\to S^{n+1}$ of degree -1. Then ρ and σ are involutive automorphisms. Let

$$\tau:\pi_{r+1}(S^{n+1};E_+,E_-)\to\pi_{r+1}(S^{n+1};E_+,E_-)$$

be the (involutive) twist automorphism defined in J2. Then in the diagram

$$\begin{array}{ccccccccc}\cdots\to\pi_r(S^n) & \xrightarrow{E} & \pi_{r+1}(S^{n+1}) & \xrightarrow{i} & \pi_{r+1}(S^{n+1};E_+,E_-) & & & & \\ \cdots\to\pi_r(S^n) & \xrightarrow{E} & \pi_{r+1}(S^{n+1}) & \xrightarrow{i} & \pi_{r+1}(S^{n+1};E_+,E_-) & \xrightarrow{\Delta} & \pi_{r-1}(S^n) & \to\cdots & \\ & & & & & \xrightarrow{\Delta} & \pi_{r-1}(S^n) & \to\cdots & \end{array}$$

(with $\sigma\downarrow$ on the middle $\pi_r(S^n)$, $\sigma\downarrow$ on the middle $\pi_{r+1}(S^{n+1})$, $\tau\downarrow$ on $\pi_{r+1}(S^{n+1};E_+,E_-)$, and $\sigma\downarrow$ on the last $\pi_{r-1}(S^n)$)

we have the relations $\sigma E=E\sigma$, $i\sigma=\rho i$, $\sigma\Delta=\Delta\rho$; $(-\sigma)E=E$, $(-\tau)i=i\sigma$, $\Delta=\Delta\tau$. Moreover, $h=h\rho$. There is a triad Whitehead product $\{\alpha,\beta\}\in\pi_{p+q+1}(S^{n+1};E_+,E_-)$, where $\alpha\in\pi_p(S^n)$, $\beta\in\pi_q(S^n)$, such that $\Delta\{\alpha,\beta\}=[\alpha,\beta]$ and $h\{\alpha,\beta\}=\alpha*\beta$; its properties are discussed extensively in J2 and J3. There is a homomorphism $\xi:\pi_{r-1}(S^{2n-1})\to\pi_{r+1}(S^{n+1};E_+,E_-)$, given by $\xi(\alpha)=\alpha^{***}\{\iota_n,\iota_n\}$, where ι_n generates $\pi_n(S^n)$ and α^{***} is the map $\pi_{2n+1}(S^{n+1};E_+,E_-)\to\pi_{r+1}(S^{n+1};E_+,E_-)$ induced by $\alpha\in\pi_{r-1}(S^{2n-1})$. Then $\Delta\xi(\alpha)=[\iota_n,\iota_n]\circ\alpha$ and $h\xi=(-1)^rE^2$.

From this rich structure, together with these relations and certain results of other authors, a host of interesting and important conclusions are drawn about homotopy groups of spheres, the suspension homomorphism, triad Whitehead products (and hence, applying Δ, about Whitehead products) and the Hopf construction. These are too numerous and detailed to be summarized, but we mention a few. Theorem (2.1): If $\alpha\in\pi_{r+1}(S^{n+1})$, then $\alpha+(-1)^n\sigma(\alpha)\in E\pi_r(S^n)$. Theorem (1.20): If $n>1$ is odd, and $E:\pi_r(S^n)\to\pi_{r+1}(S^{n+1})$, then $2E^{-1}(0)\subset E\pi_{r-1}(S^{n-1})$ and $4E^{-1}(0)=0$. Theorem (7.7): If $r<3n$, the kernel of $E:\pi_r(S^n)\to\pi_{r+1}(S^{n+1})$ consists of Whitehead products $[\alpha,\iota_n]$, $\alpha\in\pi_{r-n+1}(S^n)$. Theorem (7.8): If $r<3n$ or if $r=3n$ and $\pi_{2n+1}(S^{n+1})$ contains no element of Hopf invariant 1, then every element of $\pi_{r+1}(S^{n+1})$ is obtainable from the Hopf construction ((7.7) and (7.8) extend results of G. W. Whitehead). Corollary (1.23): The stable group of the i-stem contains no elements of order 2^{i+1}. Theorem (1.4): If $\beta\in\pi_p(S^n)$, $\gamma\in\pi_q(S^n)$, then there exists $\alpha\in\pi_{p+q-n}(S^n)$ with $\{\beta,\gamma\}=\{\alpha,\iota_n\}$; moreover, α is a natural bilinear function of β and γ. There is some overlap and interplay with results of the reviewer [J. London Math. Soc. **30** (1955), 154–172; MR **16**, 847].

The paper closes with a stimulating set of problems. E.g., if $\alpha\in\pi_{r+1}(S^{n+1};E_+,E_-)$, then $2\alpha=(-1)^n2\tau(\alpha)$. Does $\alpha=(-1)^n\tau(\alpha)$ hold? Again, if $\alpha\in\pi_r(S^n)$, n odd, then $2\alpha+2\sigma(\alpha)=0$. Does $\alpha+\sigma(\alpha)=0$ hold? Can the methods of J3 and the present paper be applied to more general Moore spaces (e.g., replace S^n by $S^n\cup e^{n+1}$)? Of a different nature is the problem of constructing a generator of the kernel of $h:\pi_{pn}(S^{n+1};E_+,E_-)\to\pi_{pn}(S^{2n+1})$, where $p>3$ is prime and n is even. The kernel is cyclic of order p and does not consist of triad Whitehead products.

P. J. Hilton (Manchester).

Weier, Joseph. Ein Satz aus der Topologie der Sphären. Collect. Math. **8** (1955–1956), 213–220.

An ordered pair of non-negative integers (n,k), $n>k$, is admissible if for each $f:S^n\to S^k$ there are maps f', f'', homotopic to f, such that $f'(x)\neq f''(x)$ for all $x\in S^n$. The author seeks admissible pairs. In this paper, he proves that (3,2) is admissible. He announces that, in a forthcoming article, he will prove (n,k) is admissible whenever $\pi_{n-1}(S^{k-1})=0$, and give some applications. {Reviewer's Remarks: The author does not observe that (n,k) is admissible whenever k is odd: set $f'=f$, $f''=rf$, r the antipodal map of S^k. Further, the theorem for (3, 2) can be proved much more simply: f and rf have the same Hopf invariant, so they are homotopic. Finally, the author's proof generalizes immediately to give his announced result on (n,k).

J. Dugundji (Los Angeles, Calif.).

Murakami, Shingo. Algebraic study of fundamental characteristic classes of sphere bundles. Osaka Math. J. **8** (1956), 187–224.

Let G be a compact Lie group, B_G a classifying space of G (base space of universal fibre bundle with group G), then the real cohomology of B_G is isomorphic with the ring $I(G)$ of invariant polynomials of G (polynomials on the Lie algebra of G invariant under the adjoint representation). The author takes this as his starting point and proceeds to study $I(G)$ algebraically. Using the first main theorem on vector invariants he recovers the known results on the structure of $I(G)$ in the cases $G=O(n)$, $SO(n)$, $U(n)$. As applications he gives proofs of the Whitney duality theorems and of the relations between Pontrjagin and Chern classes. Throughout the author is motivated by the theory of connections in differentiable fibre bundles.

M. F. Atiyah (Cambridge, England).

Weier, Joseph. Classification de champs de direction sur une sphère. C. R. Acad. Sci. Paris **243** (1956), 1995.

Let F be the set of all fields of tangent directions on a sphere S^n of dimension $n>3$. The author's theorem states

that the number of homotopy classes of F is equal to two. {If the author is adopting the common usage to mean by a tangent direction a tangent line (or a tangent 1-plane), he may have to assume that n is odd in order that F be non-empty.} *S. Chern* (Chicago, Ill.).

Weier, Joseph. On generalized vectorfields. Math. Japon. 4 (1956), 21–32.

Let M be a simply connected compact 3-manifold, and let N be a compact 2-manifold. It is proved that given a continuous $f:M\to N$ and an $\varepsilon>0$, there exists a continuous $f':M\to N$ with $d(f,f')<\varepsilon$ and $f(x)\neq f'(x)$ for all $x\in M$. It follows that if V is the space of non-zero vectors at points of N and if $p:V\to N$ is the natural fibre map, then there exists a $F:M\to V$ with $pF=f$. *E. E. Floyd.*

Guggenheimer, H. Modifications of a manifold in one point. Math. Japon. 4 (1956), 5–11.

Continuing his series of papers on topological properties of modifications [Ann. Mat. Pura Appl. (4) **41** (1956), 87–93; Bull. Res. Council Israel Sect. A. **5** (1955), 20–22; MR **17**, 1239, 1127], the author considers modifications $(\overline{M},\overline{V})$ of a manifold M at a submanifold $V\subset M$ (not necessarily complex manifolds), for which V is a single point. In this case, the cohomology algebra of $\overline{V}$ is completely determined, including multiplicative properties, and the cohomology algebra of $\overline{M}$ is determined in terms of the cohomology algebras of $\overline{V}$ of M.

R. C. Gunning (Princeton, N.J.).

Guggenheimer, H. Modifications in real and complex curves. Math. Japon. 4 (1956), 33–47.

Extending the discussion in the paper reviewed above, the author examines modifications $(\overline{M},\overline{V})$ of (M,V) for which the (real) dimension of V is either 1 or 2. In the case dim $V=1$, the cohomology algebra of $\overline{V}$ is determined and the cohomology algebra of $\overline{M}$ is expressed in terms of the cohomology algebras of M and of $\overline{V}$; in the case dim $V=2$, the cohomology groups of $\overline{V}$ are listed.

R. C. Gunning (Princeton, N.J.).

Wu, Wen-tsün. On the imbedding of polyhedrons in Euclidean spaces. Bull. Acad. Polon. Sci. Cl. III. 4 (1956), 573–577.

For any given space X, let $\tilde{X}^*$ denote the space of the ordered pairs (x_1,x_2) and X^* the space of the non-ordered pairs $\{x_1,x_2\}$, where x_1, x_2 are distinct points of X. If

$$t:\tilde{X}^*\to\tilde{X}^*,\quad \pi:\tilde{X}^*\to X^*$$

denote the maps defined by $t(x_1,x_2)=(x_2,x_1)$ and $\pi(x_1,x_2)=\{x_1,x_2\}$, then t is a homeomorphism of X^* of period 2 and X^* is the orbit space with π as the projection. Following a method of P. A. Smith, the author defines for each dimension m a cohomology class

$$\Phi^m(X)\in H^m(X^*,I_{(m)}),$$

where $I_{(m)}$ denotes the group of integers if m is even and the group of integers mod 2 if m is odd.

The main theorem of the paper states that $\Phi^m(X)=0$ is a necessary condition that X could be imbedded in the Euclidean m-space. If X is a finite polyhedron of dimension n and if $m=2n$, then the condition is also sufficient. This result includes the known results of van Kampen, Flores, Whitney, and Thom as special cases. *S. T. Hu.*

See also: Moriya, p. 636; Tannaka, p. 644; Terada, p. 644; Tashiro, p. 658; Mycielski, p. 661; McCandless, p. 662; Matschinski, p. 666; Gunning, p. 673; Dedò, p. 674.

GEOMETRY

Geometries, Euclidean and other

Schütte, Kurt. Gruppentheoretisches Axiomensystem einer verallgemeinerten euklidischen Geometrie. Math. Ann. 132 (1956), 43–62.

Following A. Schmidt [Math. Ann. **118** (1943), 609–625; MR **6**, 13], F. Bachmann [ibid. **123** (1951), 341–344; MR **13**, 767], E. Sperner [Arch. Math. **5** (1954), 458–468; **16**, 278] and H. Karzel [ibid. **6**, 66–76 (1954); 284–295 (1955); MR **16**, 395; **17**, 776], plane geometry may be introduced as follows: Let $\mathfrak{G}$ be a group generated by a system of involutoric elements $g, h, \cdots$ called lines. If gh is involutoric, then g, h are called orthogonal ($g\perp h$) and gh is called a point; points are denoted by $P, Q, \cdots$. If Pg is involutoric, P, g are called incident (P/g). The author defines an absolute reflection geometry by the following axioms: I. 1) If $P, Q/g, h$, then $P=Q$ or $g=h$; I. 2) to any P, Q there exists g such that $P, Q/g$; I.3) to any P, g there exists h such that P/h and $g\perp h$; I.4) hgh is a line; I.5) there exist three noncollinear points. These axioms differ from those given by Schmidt and Bachmann in that the "postulate of the three reflections" (if g, h, k have a common point or a common perpendicular, then ghk is a line) is replaced by the weaker axiom I.4. The reflection geometry is called euclidean if the following additional axioms are satisfied: II.1) The product of any three points is a point; II.2) any two lines have a common point or a common perpendicular. Every euclidean reflection geometry can be represented as a coordinate geometry over a quasifield; but not every quasifield leads to an euclidean reflection geometry. Precise algebraic characterizations are obtained for the cases where the geometry is restricted by the further axioms: III.1) If ghk and ghl are lines and $gh\neq hg$, then hkl is a line; III.2) if g, h, k, l are incident with the same point, then there exists m such that ghm and klm are lines.

F. A. Behrend.

Borovskiĭ, Yu. E. On the independence of the axiom of Archimedes. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 5(71), 161–167. (Russian)

The author constructs an ordered non-archimedean field U_3 which has the strong "Cantor completeness" property that every non-increasing ($\geq$) sequence of segments has a non-empty intersection. Starting with an ordered set U_1 which is constructed so that no element or gap (including the initial and terminal "gaps") has both its characters finite or denumerable, he defines first the ordered abelian group U_2 of transfinite power series with exponents in U_1 and real coefficients, and then the ordered field U_3 of transfinite power series with exponents in U_2 and real coefficients. [The "transfinite power series" method is due to Hahn, S.-B. Akad. Wiss. Wien. IIa. **116** (1907), 601–655.]

E. Mendelson.

Mendelsohn, N. S. Non-Desarguesian projective plane geometries which satisfy the harmonic point axiom. Canad. J. Math. 8 (1956), 532–562.

Verf. gibt für das vom Ref. bewiesene Resultat, das sich in einer projektiven Ebene, in der der Satz vom 4. harmonischen Punkt resp. der damit äquivalente kleine Desargues'sche Satz gilt, eine affine Koordinatengeometrie mit linearen Geradengleichungen über einen alternativen Divisionsring einführen läßt [Abh. Math. Sem. Hamburg Univ. 9 (1933), 207–222], eine Herleitung, die die Eigenschaften der Kollineationen in solchen Ebenen heranzieht. Diese Methode läßt sich möglicherweise auch beim Studium anderer nicht-Desargues'schen Ebenen mit Erfolg verwenden. — Verf. untersucht zunächst die Kollineationen in harmonischen, nicht-Desargues'schen Ebenen genauer. Die Existenz der vollen Gruppe der Schiebungen, d.h. der Homologien mit dem Zentrum auf der Achse, in einer projektiven Ebene ist äquivalent mit der Gültigkeit des kleinen Desargues'schen Satzes. Jede projektive Abbildung zwischen zwei Geraden läßt sich dann einbetten in eine Schiebung der Ebene. Ferner lassen sich dann stets 4 Urbildpunkte A, B, C, D in 4 Bildpunkte A', B', C', D', wobei A', B', C' und A, B, C kollinear sind, aber D' nicht mit A', B' und D nicht mit A, B kollinear ist, überführen durch eine in 2 Schiebungen zerlegbare Kollineation. Verf. definiert dann eine geometrische Addition $a+b$ und eine geometrische Multiplikation ab der Punkte einer Geraden g nach Ausschluß eines Punktes als unendlich ferner Punkt; er zeigt, daß diese Operationen zwischen den von ∞ verschiedenen Punkten von g einen alternativen Divisionsring bilden, wobei auf g drei Punkte 0, 1, ∞ und zwei weitere, nicht auf g gelegene Hilfspunkte P, Q, die mit dem Punkt ∞ kollinear sind, fest gewählt sind. Aus dem Satz vom 4. harmonischen Punkt folgt nur für die Operation der Addition die Unabhängigkeit von der Wahl von P, Q, aber im Gegensatz zu den Desargues'schen Ebenen nicht für die Multiplikation. Die Frage, ob sich eine Multiplikation so definieren läßt, daß das Produkt durch die Grundpunkte 0, 1, ∞ auf g bereits eindeutig bestimmt ist, führt auf die Jordan'sche Verknüpfung $a \circ b = \frac{1}{2}(ab+ba)$; bei dieser Definition der Multiplikation verliert man jedoch nach Einführung eines Koordinatensystems die Linearität der Geradengleichung. — Auf einfachen geometrischen Weg erhält Verf. den Satz, daß jeder kommutative alternative Divisionsring der Charakteristik $\neq 2$ ein Körper ist. — Nach Einführung eines affinen Koordinatensystems ergibt sich endlich in der projektiven Ebene die Gleichung der Geraden in der Form $x-y\beta-\alpha=0$ oder $y-\alpha=0$. — Abschließende Bemerkungen beziehen sich auf offene Fragen, die einerseits an nicht veröffentlichte Resultate des Verf. über ebene Geometrien in Fastkörpern anschließen, andererseits die Möglichkeit betreffen, den Fundamentalsatz über Alternativkörper geometrisch zu beweisen. *R. Moufang.*

Zaddach, Arno. Über Anti-Fano-Ebenen. Math. Z. 65 (1956), 353–388.

Als Anti-Fano-Ebene wird eine projektive Ebene bezeichnet, in der jedes Viereck kollineare Diagonalpunkte hat. In einer solchen Ebene gilt trivialerweise der Satz vom vollständigen Vierseit. Verf. beweist, daß in einer Anti-Fano-Ebene jede Unterebene, die von einem nichtentarteten Viereck und einem weiteren auf einer Viereckseite gelegenen Punkt erzeugt wird, eine projektive Ebene ist über einem Koordinatenbereich, der eine einfache Erweiterung eines Primkörpers ist. Damit überträgt Verf. das analoge vom Ref. bewiesene Resultat für projektive Ebenen, in denen das Axiom von Fano erfüllt ist und der Satz vom vollständigen Vierseit gilt, auf Anti-Fano-Ebenen [Math. Ann. **106** (1932), 755–795]. Die Frage nach der Übertragbarkeit einiger anderen vom Ref. bewiesenen Sturktursätze über solche Fano-Ebenen auf Anti-Fano-Ebenen bleibt offen. Zum Beweis seines Satzes algebraisiert Verf. die Ebene nach Ausschluß einer Geraden als unendlich ferne Gerade mit Hilfe eines Ternärkörpers nach dem Vorbild von M. Hall [Trans. Amer. Math. Soc. **54** (1943), 229–277; MR **5**, 72]. Aus dem Anti-Fano-Axiom für die Geometrie folgt dann: Jedes Element des zugehörigen Ternärkörpers genügt der Gleichung $x+x=0$; die Addition ist kommutativ und assoziativ; jedes Element des multiplikativen Loop hat ein Inverses; die Multiplikation ist potenzassoziativ. Durch wiederholte Anwendung vollständiger Induktion folgt weiter, daß in einem Anti-Fano-Ternärkörper der Bereich aller Potenzsummen eines Elementes $a \neq 0$, d. h. der Ausdrücke $\sum_{-\infty}^{+\infty} \delta_\nu a^\nu$, wobei nur endlich viel der δ_ν gleich 1, alle übrigen gleich 0 sind, ein Integritätsbereich ist. Daraus ergibt sich unmittelbar, daß jeder endliche Anti-Fano-Ternärkörper, der von einem Element erzeugt wird, ein Körper der Charakteristik 2 ist. Für den Fall, daß der Integritätsbereich der a-Potenzsummen nicht endlich ist, erfolgt die stufenweise Entwicklung der Bruchrechnung unter Zuhilfenahme vollständiger Induktion analog wie in der genannten Arbeit des Ref. und führt zu dem Ergebnis, daß der Bereich aller Brüche, die aus Potenzsummen eines Elementes mit der Addition und Multiplikation des Ternärkörpers gebildet sind, isomorph ist zu dem formal gebildeten Quotientenkörper des Integritätsbereiches aller Potenzsummen eines Elementes a. Jeder Anti-Fano-Ternärkörper, der von einem Element $a \neq 0, 1$ erzeugt wird, ist also ein Körper und zwar einfache Erweiterung des Primkörpers der Charakteristik 2. Die geometrischen Konsequenzen aus diesem Resultat sind evident.

R. Moufang (Frankfurt/Main).

Thébault, Victor. On the skew quadrangle. Amer. Math. Monthly **64** (1957), 93–96.

Two vectors **AB** and **B'A'** are divided in the same ratio. Through one dividing point a plane, orthogonal to the other vector is erected and vice versa. If O' is a point of that plane orthogonal to $A'B'$, then the necessary and sufficient condition for a point O to be in the corresponding plane, (orthogonal to AB) is $\mathbf{O'A \cdot OA' = O'B \cdot OB'}$. This is proven and applied to the four edges of a skew quadrangle $ABCD$ and the corresponding four edges of another skew quadrangle $A'B'C'D'$. The plane through the dividing point of **AB** (orthogonal to $A'B'$) and the three other planes (erected resp. for the edges of $ABCD$) being concurrent cause the analogous four planes for $A'B'C'D'$ to be concurrent too. In the particular case of all four edges being divided in the same ratio k the necessary and sufficient condition for concurrence of the four planes is developed. If $k=-1$, i.e. the dividing points are all midpoints and those four planes are concurrent, the author [Parmi les belles figures de la géométrie dans l'espace, Vuibert, Paris, 1955; MR **16**, 737] calls the two skew quadrangles orthologic relative to the midpoints of corresponding edges and develops two different formulas each giving the necessary and sufficient condition to be called thus, utilizing in the second formula $\overline{B'D'} \cdot \overline{A_1C_1} = \overline{A'C'} \cdot \overline{B_1D_1}$ the orthogonal projections A_1, C_1 of point A (resp. C) on $B'D'$ etc. whereas the first formula utilizes the eight vertices only. *R. Struik.*

Thébault, Victor. Parabolas associated with a triangle. Amer. Math. Monthly **64** (1957), 10–16.

A triangle $T\equiv ABC$ and its so-called Arzt parabolas (A), (B), (C) (characterized e.g. by AB, AC being tangents of (A) in the point B and C resp.) yield a triangle $T_0\equiv\alpha\beta\gamma$ composed of the three directrices of the parabolas. To the known relationships of T, T_0 and the parabolas [Barbarin, Mathesis **4** (1884), 142; E. Lemoine, ibid. **4** (1884), 143; H. Brocard, J. Math. Spéciales (2) **4** (1885), 104–112, 123–131 J. A. Bullard, Amer. Math. Monthly **42** (1935), 606–610] the author adds a series of theorems giving the mutual positions of orthocenter, centroid and Lemoine point of T and T_0, e.g., two altitudes of T intercept a side of T_0 so that the perpendicular bisector of this segment and the two other analogous bisectors meet in one point, the midpoint of $\overline{GH}$ (G is the centroid, H the orthocenter of T). These theorems are productive of metric relationships, e.g., the distances of G to the directrices are inversely proportional to the length of the medians of T, and so are the parameters of the parabolas. The exposition is accompanied by an elucidating figure of aesthetic beauty. *R. Struik* (Cambridge, Mass.).

Kelly, Paul; and Straus, Ernst. A characteristic property of the circle, ellipse, and hyperbola. Amer. Math. Monthly **63** (1956), 710–711.

It is well-known that a conic has the same pedal curve with respect to either focus. The authors prove: if a piecewise differentiable curve C has the same pedal curve with respect to two points p_1 and p_2, then every arc of C belongs to a conic whose foci are p_1 and p_2. *O. Bottema.*

Deaux, R. Sur les ellipses tritangentes à l'hypocycloïde de Steiner d'un triangle. Mathesis **65** (1956), 526–534.

Let ABC be a triangle with circum-radius R, circum-centre O, centroid G, and orthocentre H. Let P be a variable point on the circum-circle, and let P_a, P_b, P_c be the orthogonal projections of P on the three sides. These three points lie on a line (the Simson line) whose envelope is Steiner's 3-cusped hypocycloid, H_3. The author considers the system of ellipses that are tritangent to H_3 while their centres lie on the Euler line OGH. He shows that the ellipse with centre G has axes $|R\pm OG|$ and is the locus of the centroid of the three points P_a, P_b, P_c; moreover, the normals at its points of contact with H_3 all pass through O. He discusses also the ellipse with centre O, whose axes are $|R\pm OH|$, and the ellipse with centre H, which passes through the feet of the altitudes.

H. S. M. Coxeter (Toronto, Ont.).

Matschinski, Matthias. Polytopes réguliers des séries du cube (PC) et du décahexaèdre (PD). Exemples de polytopes: a. contenant un espace „euclidien" fermé et b. correspondant à un espace à courbure constante non fermé. C. R. Acad. Sci. Paris **244** (1957), 717–720.

As the author remarked in his earlier paper [same C. R. **243** (1956), 472–475; MR **18**, 227], the regular skew polyhedron $\{4,4|p\}$ is formed by the vertices, edges and squares of the four-dimensional "prism" $\{p\}^2=\{p\}\times\{p\}$: the rectangular product of two regular p-gons, $\{p\}$ [Coxeter, Regular polytopes, Methuen, London, 1948, p. 124; MR **10**, 261]. This now appears as the case $n=2$ of the regular skew polytope formed by the elements γ_m $(m=0, 1, \cdots, n)$ of the $2n$-dimensional prism $\{p\}^n$: the rectangular product of n regular p-gons. Topologically the general case is the n-dimensional torus derived from the n-dimensional cubic honeycomb by reducing the coordinates modulo p. The author (who unnecessarily restricts attention to the case $p=4$) observes that the Euler-Schäfli characteristic $N_0-N'+N_2-\cdots$ is always zero, instead of being $1+(-1)^n$ as in the case of an ordinary polytope with n-dimensional cells in $(n+1)$-space [Coxeter, op. cit., p. 163]. The agreement when n is odd does not mean that the skew polytope is then simply-connected. *H. S. M. Coxeter* (Toronto, Ont.).

Barlotti, Adriano. Sui $\{k; n\}$-archi di un piano lineare finito. Boll. Un. Mat. Ital. (3) **11** (1956), 553–556.

Sei S_q eine endliche Ebene über einem Galoisfeld der Ordnung q und sei ein (k, n)-Bogen eine Menge von k Punkten aus S_q, von denen niemals $n+1$ Punkte auf einer Geraden liegen. Die Frage nach dem größten Wert von k zu dem bei gegebenem n ein (k, n)-Bogen in S_q existieren kann, wird beantwortet durch den mit Hilfe abzählender Betrachtungen bewiesenen Satz: Für $2<n\leqq q$ gibt es in S_q einen (k, n)-Bogen nur dann, wenn $k\leq (n-1)q+n-2$ im Falle $q\not\equiv 0 \pmod n$ beziehungsweise $k\leq (n-1)q+n$ im Falle $q\equiv 0 \pmod n$. — Ein vollständiger (k, n)-Bogen ist ein solcher, der nicht in einem (k', n)-Bogen mit $k'>k$ enthalten ist. Für $q\equiv 0 \pmod n$ existieren keine vollständigen (k, n)-Bögen mit $k=(n-1)(q+1)$. [Vergl. insbesondere B. Segre, Ann. Mat. Pura Appl. (4) **39** (1955), 357–379; MR **17**, 776.] *R. Moufang* (Frankfurt/Main).

Hughes, D. R. Regular collineation groups. Proc. Amer. Math. Soc. **8** (1957), 165–168.

Let v, k, λ $(0<\lambda<k<v)$ be integers such that $\lambda(v-1)=k(k-1)$. Suppose π is a collection of v points and v lines such that every point (line) is on k lines (contains k points), and such that every pair of distinct points (lines) are on λ common lines (contain λ points in common). Then we call π a λ-plane. If ϕ is a one-to-one mapping of π, sending points onto points and lines onto lines, and preserving incidence, we call ϕ a collineation of π. If π is a λ-plane possessing a collineation group G of order m such that no non-identity element of G fixes any point or line of π, we say that π is regular of degree m (with group G). The author shows that regularity implies the existence of a matrix relation similar to the well-known relations involving incidence matrices [see Bruck and Ryser, Canad. J. Math. **1** (1949), 88–93; MR **10**, 319; Chowla and Ryser, ibid. **2** (1950), 93–99; MR **11**, 306] and, in fact, include these relations as special cases. Also, the theorems of the author prove, for a wide class of integers n, that no projective plane of order n can be regular of degree greater than one.

Of the 4 theorems proved by the author we cite the last two: Theorem 3. If π is a regular λ-plane of degree m, with parameters v, k, λ, then there is a square matrix A of order $t=v/m$, consisting entirely of non-negative integral entries, such that $A^TA=AA^T=B$, where B has $n+\lambda m$ on the main diagonal and λm elsewhere. Furthermore, every row or column sums to k. Here $n=k-\lambda$. Theorem 4. If π is a regular λ-plane of degree m with the parameters v, k, λ, and if $t=v/m$ is odd, then the equation (again, $n=k-\lambda$) $x^2=ny^2+(-1)^{(t-1)/2}\lambda mz^2$ has a non-trivial solution in integers. *S. Chowla.*

Basotti, Lucilla. Sulle relazioni a tre termini fra le coordinate di Grassmann. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 200–204, 318–325.

As it is well-known, Grassmann coordinates of a subspace S_k of a projective space S_n satisfy a number of

quadratic relations (representing the Grassmann manifold $G(n, k)$ of $S_k CS_n$). Among these a certain number of three-term relations exist. It has been assumed for a long time that the three-term relations are equivalent to the full set of relations. B. Segre [Lezioni di geometria moderna, v. 1, Zanichelli, Bologna, 1948; MR **10**, 729] proved the contrary by an example.

Bompiani [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **13** (1952), 329–335; MR **14**, 1014] gave the necessary and sufficient conditions that skew-symmetric coordinates satisfying the three-term relations represent a point $\epsilon G(n, k)$.

The author studies the manifold $F(n, k) \supset G(n, k)$ represented by the three-term relations making use of Bompiani's theorem.

A complete classification of the components of $F(n, 2) \neq G(n, 2)$ is given, determining their irreducible components (their number, their dimensionalities, their orders, their collineation groups).

Some general results are also given for $k>2$; whereas $G(n, 2)$ is the component of the largest dimensionality of $F(n, 2)$, for $k>2$ and $n \geqq k(k+2)$ there are component of $F(n, k)$ whose dimensionality is greater than that of $G(n, k)$. *E. Bompiani* (Rome).

Klingenberg, Wilhelm. Projektive Geometrien mit Homomorphismus. Math. Ann. **132** (1956), 180–200.

A collection $\mathfrak{P}_3$ of points in which lines and planes are distinguished subsets is called a projective space with homomorphism if $\mathfrak{P}_3$ possesses an incidence preserving mapping onto a (3-dimensional) projective space $\overline{\mathfrak{P}}_3$ (in the usual sense) such that the usual axioms of projective spaces are satisfied in $\mathfrak{P}_3$ whenever they are satisfied by the corresponding images in $\overline{\mathfrak{P}}_3$. Similarly, a projective plane with homorphism is defined. An algebraic description of projective spaces with homomorphism is given: $\mathfrak{P}_3$ is coordinatized (in a fashion analogous with classical "homogeneous coordinates") by a ring A with identity and an ideal I which contains all proper left and right ideals of A; the division ring coordinatizing $\overline{\mathfrak{P}}_3$ is isomorphic to A/I. In the special case that $\mathfrak{P}_3$ is itself a projective space, the ring A contains no zero divisors and for every $a, b \epsilon A$, there are $x, y \epsilon A$ such that $ax=b$ or $bx=a$, and $ya=b$ or $yb=a$; furthermore, the (left or right) quotient division ring of A is the coordinatizing division ring of $\mathfrak{P}_3$. Similar results are obtained for projective planes with homomorphism, where it is assumed that the plane possesses a transitive translation group for some choice of a line at infinity.

D. R. Hughes (Columbus, Ohio).

Arnese, Giuseppe. Classificazione dei fasci di complessi lineari di rette in S_7. Rend. Mat. e Appl. (5) **15** (1956), 119–128.

Bundles of $\infty^1 S_2$-complexes in S_7 (7-dimensional projective space) are considered using a method of C. Longo [Rend. Sem. Mat. Univ. Padova **24** (1955), 300–311; MR **17**, 886]. The canonical form of reciprocity can be written in the form $u_\rho' = (\lambda a_{\alpha\rho} + \mu b_{\alpha\rho}) x^\alpha$ ($\alpha=0, \cdots, 3$; $\rho=4, \cdots, 7$). Now, the general reciprocity may be non-degenerate A, or simple B, double C, or triple D degenerate one, twice or thrice. These sections can be subdivided. For A we get $1+2+3+3+3=12$ cases, for B: $2+4+2+1+2+1=12$ cases and for C: $3+2+1=6$ cases. For every case the canonical form is given.

J. A. Schouten (Epe).

Bruins, E. M. Orthogonal transversals on the tetrahedron in non-euclidean geometry. Bull. Coll. Arts Sci., Baghdad **1** (1956), 18–23.

It is shown that in non-euclidean geometry, there exist eight points from which three mutually perpendicular transversals over the pairs of opposite edges of a tetrahedron can be drawn. This is done using the symbolic method of invariant theory for the general case. In the special case of a "regular tetrahedron" the result is deduced also in a non-symbolic way. (Author's summary.)

L. A. Santaló (Buenos Aires).

Širokov, A. P. The geometry of generalized biaxial spaces. Kazan. Gos. Univ. Uč. Zap. **114**, no. 2 (1954), 123–166. (Russian)

A biplanar space B_{2n+1} is a real projective space of odd dimensions in which two skew n-dimensional planes are given as the absolute. The group is that subgroup of the projective space which leaves the absolute invariant. If the absolute consists of real planes the B_{2n+1} is called hyperbolic, if they are conjugate complex, the B_{2n+1} is elliptic. For $n=1$ we obtain the biaxial spaces of A. P. Norden. A B_{2n-1} can be taken as a local pseudovector field of an affine A_{2n} with two not intersecting simple n vectors $v^{\alpha_1 \cdots \alpha_n}$, $w^{\alpha_1 \cdots \alpha_n}$. In the B_{2n-1} indices are raised and lowered with the aid of $\varepsilon_{\alpha_1 \ldots \alpha_{2n}}$ and $\varepsilon^{\beta_1 \cdots \beta_{2n}}$. There is an absolute involution by which each point x is associated with harmonic point with respect to the points of intersection its with the absolute of the only line through x which intersects the absolute. The tensor $g_\beta{}^\alpha$ of the absolute involution can be found from

$$\frac{1}{n} v^{(\sigma)} w_{(\sigma)} g_\alpha{}^\beta = v^{\beta\sigma_2 \cdots \sigma_n} w_{\alpha\sigma_2 \ldots \sigma_n} - (-1)^n w^{\beta\sigma_2 \cdots \sigma_n} v_{\alpha\sigma_2 \ldots \sigma_n},$$

$$g_\alpha{}^\sigma g_\sigma{}^\beta = \delta_\alpha{}^\beta,$$

in hyperbolic space, in elliptic space, where $w^{(\sigma)} = \bar{v}^{(\sigma)}$, the corresponding equation is

$$\frac{i}{n} v^{(\sigma)} \bar{v}_{(\sigma)} g v^\beta = v^{\beta\sigma_2 \cdots \sigma_n} \bar{v}_{\alpha\sigma_2 \ldots \sigma_n} - (-1)^n \bar{v}^{\beta\sigma_2 \cdots \sigma_n} v_{\alpha\sigma_2 \ldots \sigma_n},$$

$$g_\alpha{}^\sigma g_\sigma{}^\beta = -\delta_\alpha{}^\beta.$$

Most of the investigation deals with the elliptic type. A connection is determined in the A_{2n} for which $\nabla_\beta v^{\alpha_1 \cdots \alpha_n} = \lambda_\beta v^{\alpha_1 \cdots \alpha_n}$, $\nabla_\beta w^{\alpha_1 \cdots \alpha_n} = \mu_\beta w^{\alpha_1 \cdots \alpha_n}$, then $\nabla_\gamma g_\beta{}^\alpha = 0$. The resulting geometry is utilized for the geometry of normalized hypersurfaces. For the rich details we must refer to the original paper. *D. J. Struik* (Cambridge, Mass.).

See also: Meinardus, p. 642; Goldberg, p. 668; Hollingsworth and Bell, p. 669; Wilson, p. 686; Rogers, p. 686; Rogers and Wilson, p. 687; Rogers, Stanley and Wilson, p. 687; Loedel, p. 703; Eulerus, p. 709.

Convex Domains, Integral Geometry

Viet, Ursula. Umkehrung eines Satzes von H. Brunn über Mittelpunktseibereiche. Math.-Phys. Semesterber. **5** (1956), 141–142.

Let E be a bounded convex domain; S=center of gravity of E. Theorem 1: There are three secants of E bisected by S. Theorem 2: Suppose the boundary of E contains no straight line segments. Then E has a midpoint if and only if no point $\neq S$ bisects more than one secant of E. The necessity of this condition was proved by H.

Brunn [Thesis, Munich, 1887]. A three-dimensional analogue of Theorem 2 is also given. *P. Scherk.*

Matérn, Bertil. On the geometry of the cross-section of a stem. Medd. Statens Skogsforskningsinst. **46** (1956), no. 11, 28 pp. (Swedish summary)

Some common methods of estimating the area of a cross-section of a tree stem are discussed. General results are obtained by aid of well-known methods of the theory of convex curves and integral geometry. Use is especially made of the Fourier-development of the supporting function of a convex region. *Author's summary.*

Mahler, Kurt. Über die konvexen Körper, die sich einem Sternkörper einbeschreiben lassen. Math. Z. **66** (1956), 25–33.

1. In the affine space R_n a point set S is called a "Sternkoerper" (star-shaped body) if tX, $-1<t<1$ is an interior point of S for all $X \subset S$. There exists a convex body K, inscribed in S, of maximum volume $V(K)$. 2. Let $X_1 \cdots X_n$ form a lattice defined by $\sum u_i X_i$, where the u_i are integers and let $\Delta(S)$ be the lower limit of the determinants of those lattices which have the property that none of their points is an interior point of S. 3. The author proves the theorem that to any arbitrarily small ε there exists a "Sternkoerper" S in R_n satisfying the inequality (1) $V(K)<\varepsilon\Delta(S)$. *H. Knothe.*

Pleijel, Arne. Zwei kurze Beweise der isoperimetrischen Ungleichung. Arch. Math. **7** (1956), 317–319.

Let $s(\varphi)$ be the arc length of a closed convex curve K measured from a fixed origin to the contact point (assumed unique) of the line of support corresponding to the direction φ. If u, v are the angles between the respective tangents to K and the chord which unites the points $s(\varphi)$, $s(\alpha)$ the formula

$$2\pi F = \int_0^{2\pi} ds(\alpha) \int_\alpha^{\alpha+\pi} \sin u \sin v \, ds(\varphi)$$

holds and from it and others of the same kind the isoperimetric inequality $U^2-4\pi F \geq 0$ is easily deduced (U=length, F=area enclosed by K). *L. A. Santaló.*

Goldberg, Michael. Trammel rotors in regular polygons. Amer. Math. Monthly **64** (1957), 71–78.

Continuing his investigation of circular-arc rotors in a regular plane n-gon [same Monthly **55** (1948), 393–402; MR **10**, 205], the author describes such a curve having the symmetry of a regular $(n\pm1)$-gon. It is composed of arcs of circles of $[\frac{1}{2}(n-1)]$ different radii; e.g., when $n=4$, it is composed of 5 or 3 equal arcs. He conjectures that the latter variety (having $(n-1)$-gonal symmetry) is the rotor of least area in the given n-gon. He obtains similar results for a spherical n-gon [J. Math. Phys. **34** (1956), 322–327; MR **17**, 655], except that the arcs of circles are replaced by arcs of twsited curves which are loci of points on the sphere from which two fixed points subtend constant angles. Finally, he mentions the problem of solid rotors in a regular polyhedron.

H. S. M. Coxeter (Toronto, Ont.).

Masotti Biggiogero, Giuseppina. Sulla geometria integrale: Generalizzazione di formule di Crofton, Lebesgue e Santalo. Rev. Un. Mat. Argentina **17** (1955), 125–134 (1956).

Let $f(\omega)$ be a fonction for which the integrals

$$h=\int_0^\pi \frac{f(\omega)}{\sin\omega}d\omega, \quad k=\int_0^\pi \frac{f(\omega)}{1-\cos\omega}d\omega$$

exist and are finite. Let T, T_1 be the lengths of the tangents to an oval K of length L from an exterior point P, ω the angle between these tangents and R, R_1 the radii of curvature of K at the points of contact. Then the following integral formulas are shown to hold:

$$\int \frac{f(\omega)}{TT_1}dP=2\pi h, \quad \int \frac{f(\omega)}{TT_1}(R+R_1)dP=2hL,$$

$$\int f(\omega)\left(\frac{1}{T}+\frac{1}{T_1}\right)dP=2kL,$$

where dP is the element of area at P and the integrals are extended over the exterior of K. By particular choices of $f(\omega)$ some known integral formulas are obtained.

L. A. Santaló (Buenos Aires).

See also: Nef, p. 659; Bauer, p. 660; Collatz, p. 677; Eulerus, p. 709.

Differential Geometry

Lane, N. D. Differentiable points of arcs in conformal n-space. Pacific J. Math. **6** (1956), 301–313.

Let $P, \cdots [S^{(m)}, \cdots]$ denote points [m-spheres] in conformal n-space. An arc A is the continuous image of an interval. The letters p, t denote both points on A and their parameters. Let $1\leq m<n$. If p, $P_1, \cdots, P_{m+1}$ do not span an $S^{(m-1)}$, there is a unique m-sphere $S^{(m)}(P, \cdots, P_{m+1}; \tau_0)$ through these points. The arc A is called $(m+1)$-times differentiable at p if the following sequence of conditions $\Gamma_1^{(m)}, \cdots, \Gamma_{m+1}^{(m)}$ is satisfied. $\Gamma_k^{(m)}$: If the parameter t is sufficiently close to p, $t\neq p$, then the m-sphere $S^{(m)}(P_1, \cdots, P_{m+1-k}, t; \tau_{k-1})$ is uniquely defined. It converges if $t\to p$. Denote its limit $S_k^{(m)}=S^{(m)}(P_1, \cdots, P_{m+1-k}; \tau_k)$. — Write $S_{m+1}^{(m)}=\lim_{t\to p} S^{(m)}(t; \tau_m)$ and put $S_1^{(0)}=p$.

Condition $\Gamma_1^{(m)}$ implies all the $\Gamma_k^{(\lambda)}$ except possibly $\Gamma_{m+1}^{(m)}$; $1\leq k\leq\lambda+1\leq m+1$. Thus $(m+1)$-times differentiability implies m-times differentiability; $\Gamma_1^{(n-1)}$ implies $(m+1)$-times differentiability $(m+1<n)$; and p is n-times differentiable if and only if $\Gamma_1^{(n-1)}$ and $\Gamma_n^{(n-1)}$ hold. If $S_m^{(m-1)}\neq p$, $\Gamma_1^{(m)}$ implies $\Gamma_{m+1}^{(m)}$ and hence $(m+1)$-times differentiability. — Let $1\leq k\leq m<n$. Assume $\Gamma_1^{(m)}$. If $S_k^{(k-1)}\neq p$, the family $\tau_k^{(m)}$ of all $S_k^{(m)}$ consists of the $S^{(m)}$ through $S_k^{(k-1)}$. If $S_k^{(k-1)}=p$, choose any $S_k^{(k)} \subset t_k^{(k)}$. Then $\tau_k^{(m)}$ consists of the $S^{(m)}$ touching $S_k^{(k)}$ at p. Obviously $\tau_0^{(m)} \supset \tau_1^{(m)} \supset \cdots \supset \tau_{m+1}^{(m)}$. — Let A be n-times differentiable at p. Then there is an index i with $1\leq i\leq n$ such that $S_{k+1}^{(k)}=p$ if and only if $k<i$. From the above, i determines the structure of the lattice of the $\tau_k^{(m)}$'s.

Let p be an n-times differentiable interior point of A. Then every $S^{(n-1)}\neq S_k^{(n-1)}$ either supports or intersects A at p [the point-sphere p supports by definition]. All the $S^{(n-1)}$ which belong to $\tau_k^{(n-1)}$ but not to $\tau_{k+1}^{(n-1)}$ support, or they all intersect; $k=0, 1, \cdots, n-1$. If $k=i$, they support. This leads to a classification of the n-times differentiable points into $(3n-1)\cdot 2^{n-1}$ types. All of them are shown to exist. *P. Scherk* (Philadelphia, Pa.).

★**Favard, J. Cours de géométrie différentielle locale.** Gauthier-Villars, Paris, 1957. x+553 pp. 6000 francs.

Differential geometry has advanced very rapidly in the last decades: it is time, states the author, to reflect on its principles, to point out its method and to make a reasonable selection of old and new material. Even local differential geometry has still many problems of direct

and actual importance to offer, it has even curious gaps (such as cinematic geometry of more than two parameters), moreover, its foundations are still little studied. This book is thus an attempt to take these factors into consideration; while modernizing the instruction, it does not aim at too much of a break with tradition and presents "un caractère de transition". There is much here which belongs to present day research, although the classical theorems are all there, even if some of them are relegated to the exercises. The result of the author's endeavor of reconciliating past and present is a book of rich and varied content, by the study of which both beginner and mature scholar can greatly benefit. — The opening chapters, which form the introduction, lead us into the notions of topological, compact and metrical spaces, topological groups and Lie groups, as well as the groups of differential geometry; this is followed by affine tensor algebra and Cartan's calculus of exterior forms. Then, in the First Part, called "Direct infinitesimal geometry" come generalities on immersed manifolds and parametrisation in manifolds of n dimensions, together with the theory of contact, of envelopes (carefully presented, even if, as the author writes, the foundations of the theory of envelopes are not yet in a quite satisfactory state), and of contact transformations. The Second Part, entitled "Classical geometries", contains a First Section, which in four chapters presents the ordinary differential geometry of curves and surfaces in considerable detail, and in a special chapter "Ruled euclidean geometry" with ruled surfaces, line congruences and complexes. The Second Section has unimodular affine differential geometry, the Third Section projective differential geometry of curves and surfaces in two- and three-space. The Third Part, named "Transport theory" deals with parallel displacement; it presents the geometry of spaces of affine and of Riemannian connection and ends with the theory of manifolds imbedded in Riemannian manifolds. There are a number of exercises, many of which are quite interesting, some contain little known theorems. As chief method of exposition the author has selected the form in which E. Cartan has presented so many of his results.

D. J. Struik (Cambridge, Mass.).

Hollingsworth, B. J.; and Bell, P. O. Generalized conjugate nets in projective n-space. Duke Math. J. 23 (1956), 601–607.

Im Anschluss an einige Arbeiten des zweiten Verfassers [Trans. Amer. Math. Soc. **60** (1946), 22–50; Proc. Amer. Math. Soc. **3** (1952), 300–302; Duke Math. J. **21** (1954), 323–327; MR **7**, 529; **13**, 775; **16**, 168] werden die Sätze von Darboux über dreifache orthogonale Flächensysteme und über Systeme von Parallelflächen im R_3 [Darboux, Leçons sur les systèmes orthogonaux et les coordonnées curvilignes, 2ième éd., Gauthier-Villars, Paris, 1910], sofern sie sich auf Aussagen über konjugierte Netze beziehen und projektiven Charakter haben, mit analytischen Methoden auf den projektiven R_n verallgemeinert. Bei dem letztgenannten Satz ist der Darboux'sche Begriff paralleler Flächen zu ersetzen durch den Begriff perspektiver Hyperflächen: zwei Hyperflächen heissen perspektiv bezüglich einer festen Hyperebene π, wenn sich die Tangentialhyperebenen in entsprechenden Punkten in einer $(n-2)$-Ebene von π schneiden. Dann gilt: Ist G_n eine Hyperkongruenz, die konjugiert ist zu einer Hyperfläche V_0, so existieren unendlich viele zu G_n transversale Hyperfläachen, die perspektiv sind zu V_0 bezüglich einer festen Hyperebene π. Dieser Satz lässt noch eine Verallgemeinerung zu.

R. Moufang (Frankfurt).

Cîmpan, Florica. Surfaces à courbure Bacaloglu constante. Rev. Univ. "Al. I. Cuza" Inst. Politehn. Iaşi 2 (1955), 75–85. (Romanian. Russian and French summaries)

An S_B is a surface in the euclidean 3-space for which the expression:

$$\frac{1}{R_1^2}+\frac{2}{3R_1R_2}+\frac{1}{R_2^2},$$

called "Bacaloglu curvature", is constant. R_1 and R_2 denote the principal radii of curvature. The author derives α) a canonical form for the line element of n S_B by referring it to its line of curvature, β) the condition for an S_B to be applicable on a surface of revolution, and γ) the equations of surfaces S_B which are in the same time of revolution.

The following theorem is proved: If from the points of an S_B one measures in the same direction on the normals the distance R_1+R_2, the surface thus obtained is also an S_B.

R. Blum.

Pinl, M. Zur Integration der isotropen Komplexe in R_5. Monatsh. Math. **60** (1956), 298–312.

The full-isotropic planes in R_5 (§ 1) are used in § 2 for the general isotropic curves in R_5, and in § 3 for the hyperspherical and twofold isotropic curves in R_5. In both cases the analytic functions used must satisfy certain conditions that are discussed in § 4. § 5 deals with Hilbert's solutions of Monge's problem. *J. A. Schouten.*

Kerawala, Sulaiman. On the locus of the centre of spherical curvature. Ganita **6** (1955), 23–26 (1956).

Consider the curves C, C_1, C_2, where each is the locus of the center of spherical curvature of the preceding one. Let s, ρ^{-1}, σ^{-1} be the arc length curvature, and torsion of C. Jha pointed out that C_2 and C will be identical if ρ and σ satisfy a certain differential equation [Ganita **4** (1953), 131–134; MR **16**, 399]. The present author rewrites that equation in the form

$$(1)\quad 2v\frac{d^4\rho}{dx^4}+5\frac{dv}{dx}\frac{d^3\rho}{dx^3}+2\Big(2\frac{d^2v}{dx^2}+v+1\Big)\frac{d^2\rho}{dx^2}+\Big(\frac{d^3v}{dx^3}+\frac{dv}{dx}\Big)\frac{d\rho}{dx}=0,$$

where $x=\int^s\rho^{-1}\,ds$ and $\rho\sigma^{-1}=v^{-\frac{1}{2}}$. Then if $\rho=u(x)$ is chosen arbitrarily, and if (1) is solved for $v(x)$ one has C given in parametric form: $\rho=u(x)$, $\sigma=u(x)\sqrt{v(x)}$ with $s=\int^x u(x)dx$. This is done for the particular cases $u(x)=a$, $2(ax+b)$, ce^{ax}, $a\log x$, $3(ax^2+bx+c)$. Alternatively, one can choose $v=v(x)$ arbitrarily and try to solve (1) for ρ. This is also done for several particular cases.

A. Schwartz.

Grabiel, Federico. Global differential geometry in physical measurements. Rev. Un. Mat. Argentina **17** (1955), 69–71 (1956). (Spanish)

Özkan, Asim. Les surfaces W (Weingarten) à lignes de courbure planes dans les deux systèmes. Arch. Math. **7** (1956), 386–390.

The author discusses the problem of determining the Weingarten surfaces or the surfaces W in the class of surfaces for which their lines of curvature are all plane

curves. If R and $\bar{R}$ denote the principal radii of curvature of a surface S, then it is a surface W if and only if

$$\frac{\partial R}{\partial u}\frac{\partial \bar{R}}{\partial v}=\frac{\partial R}{\partial v}\frac{\partial \bar{R}}{\partial u},$$

where (u, v) denote curvilinear coordinates on S. O. Rozet, H. Paquet, and N. Legrain-Pissard showed that the subclass of such surfaces for which $R+\bar{R}=$constant, consists of surfaces W [Bull. Soc. Roy. Sci. Liège **18** (1949), 259–265, 343–346; MR **11**, 394; **12**, 203]. The author demonstrates that in the set of surfaces W for which their lines of curvature are all plane curves, there are surfaces for which $R+\bar{R}$ is not constant. *J. De Cicco.*

Barner, Martin. Kinematische Fragen in der Projektivgeometrie der Regelflächen. Monatsh. Math. **60** (1956), 21–56.

In a three dimensional projective space R_3, a curve is called a complex curve if its tangents belong to a linear complex, and a surface a complex surface if all members of one family of its asymptotic curves are complex curves. Associated with a ruled surface R in R_3 and two asymptotic curves Γ_1, Γ_2 of R, there is an invariant linear congruence C, which is a W-congruence with complex focal surfaces. This congruence C originates from a complex ruled surface under a projective motion. The paths of this motion describe a one-parameter family of complex ruled surfaces projective to each other, and establish a line-wise correspondence between the complex ruled surfaces and the ruled surface R. Under this correspondence all asymptotic curves of R correspond to each other, and also the invariants $\mathfrak{a}$, α_1, α_2 of R with respect to Γ_1, Γ_2 correspond to those of the complex ruled surfaces. For the complex ruled surfaces the invariant α_0 vanishes. Among other things in the kinematic theory of ruled surfaces, the smallest number of turning points of a closed asymptotic curve of a closed ruled surface and a mapping of a ruled surface into a family of lines of a quadric are discussed. Finally, an outline of the corresponding results in Lie's geometry of spheres is given.
C. C. Hsiung (Bethlehem, Pa.).

Bhattacharya, P. B.; and Ram, Behari. Congruences of Guichard. Ganita **5** (1954), 237–249 (1955).

A congruence of Guichard is a system of ∞^2 lines in space such that the focal surfaces are intersected by the developable surfaces in the lines of curvature. The authors obtain the analytic conditions that a system of contravariant vectors $p_1{}^a$ on a surface S_1 determines a congruence of Guichard with the given surface S_1 as one of the focal surfaces. When the $p_1{}^a$ are the contravariant components of the tangent vector to a simple family of the lines of curvature which are parametric on S_1, the general equations reduce to known results. As a consequence of this work, the known result is proved that when a congruence consist of the tangents to one system of lines of curvature on a surface, the focal distances are equal to the radii of geodesic curvature of the other system of lines of curvature. New proofs are given of the following known theorems. (I) The necessary and sufficient conditions that a congruence be that of Guichard are that the spherical representations of the developables of the congruence are the same as those of the asymptotic lines on a pseudospherical surface. (II) A necessary and sufficient condition that tangents to one system of lines of curvature on a surface form a congruence of Guichard is that the evolute of the surface corresponding to the same system of lines of curvature form a surface of Voss. (III) A plane, normal to a ray L of a congruence of Guichard, and which divides the segment between the focal points of L in a constant ratio, envelopes a surface of Voss. In particular, the middle envelope of a congruence of Guichard is a surface of Voss. *J. De Cicco* (Chicago, Ill.).

Gasapina, Umberto. Sull'inverso del teorema di Reiss. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. **90** (1956), 348–355.

Reiss proved that the 2nd order elements E_2 of a plane algebraic curve of order n, C^n, having their centers on a line must satisfy a certain relation. The sufficiency of this condition for the belonging of n arcs to a C^n has been proved by Engel. A new proof is given here using the Euclidean space. Differential consequences of this condition are also given. *E. Bompiani* (Rome).

Fernández, Germán. Affine differential geometry of hypersurfaces. Rev. Un. Mat. Argentina **17** (1955), 29–38 (1956). (Spanish)

The author gives the essentials of the affine differential geometry of hypersurfaces by means of the notation and methods of E. Cartan. After the equations of Frenet, the following integral formula is proved. Let H_k be the affine mean curvature of order k ($k\leq n-1$) of a closed orientable hypersurface S at a point P, $d\Omega$ the element of affine area at P and w the affine distance from a fixed point 0 contained in the interior of S to P. Then

$$\int_S H_k d\Omega=\int_S H_{k+1} w d\Omega,$$

a formula which generalizes to the affine geometry a well-known result of the metric case. *L. A. Santaló.*

See also: Gallo, p. 653; Gallarati, p. 672.

Riemannian Geometry, Connections

See: Kohn and Spencer, p. 658; Murakami, p. 663; de Simoni, p. 684; Horváth, p. 704.

Complex Manifolds

Dolbeault, Pierre. Formes différentielles et cohomologie sur une variété analytique complexe. I. Ann. of Math. (2) **64** (1956), 83–130.

Ce travail tire son origine d'une étude des formes différentielles méromorphes sur une variété analytique complexe, qui sera développée dans une Partie II. Les problèmes concernant le passage du local au global sont traités par les méthodes cohomologiques. Dans une Introduction on rappelle: 1° quelques résultats essentiels de la théorie cohomologique des faisceaux; 2° quelques propriétés des variétés analytiques complexes, notamment des variétés kählériennes compactes et des variétés de Stein. Le Chapitre I contient une démonstration détaillée de deux théorèmes déjà annoncés [C. R. Acad. Sci. Paris **236** (1953), 175–177; MR **14**, 673] et souvent utilisés depuis par les spécialistes: le premier théorème de Dolbeault établit un isomorphisme natural entre $H^{p,q}(V)$ (espace vectoriel de $\bar{\partial}''$-cohomologie de type (p, q) de la variété analytique complexe V) et $H^q(V, \Omega^p)$ (cohomologie de dimension q de V à coefficients dans le faisceau Ω^p

des p-formes différentielles holomorphes); le deuxième théoréme établit un isomorphisme naturel entre $K^{p,q}(V)$ (espace des éléments de degré $p+q$ de la d-cohomologie des formes différentielles — ou des courants — dont le premier type est $\geqq p$), et $H^q(V, E^p)$, où E^p désigne le faisceau des formes différentielles holomorphes fermées de degré p. Des relations précises sont établies entre ces divers espaces de cohomologie; le cas des variétés kählériennes compactes et des variétés de Stein est explicité. L'espace $K^{p,1}(V)$ intervient dans le problème de la recherche d'une p-forme méromorphe ayant des „parties singulières" données.

Dans le Chapitre II, on étudie d'abord les obstructions topologique et analytique d'un espace fibré analytique principal dont le groupe structural est le groupe multiplicatif C^* des nombres complexes $\neq 0$. L'élément de $H^2(V, Z)$ qui correspond à un diviseur de V s'interprète, on le sait, en utilisant la „dualité" des variétés, par le moyen du $(2n-2)$-cycle fondamental porté par le diviseur; cette interprétation est, pour la première fois dans la littérature, semble-t-il, démontrée explicitement dans le cas le plus général où la sous-variété portant le diviseur peut avoir des singularités. Enfin, le problème des „fonctions theta" est étudié comme cas particulier de celui des p-formes méromorphes „additives", c'est-à-dire des p-formes méromorphes non uniformes qui, par un lacet de V, se trouvent augmentées d'une forme holomorphe et uniforme. La méthode donnée couvre à la fois le cas des variétés kählériennes compactes (Kodaira) et celui des polycylindres (Stein). *H. Cartan* (Paris).

See also: Stein, p. 649; Kohn and Spencer, p. 658; Guggenheimer, p. 664.

Algebraic Geometry

Rosina, Bellino Antonio. Sul numero dei circuiti dispari delle curve algebriche reali situate sopra superficie algebriche d'ordine dispari prive di singolarità. Boll. Un. Mat. Ital. (3) **11** (1956), 419–421.

A real algebraic curve without singularities, lying on a real algebraic surface of odd order, likewise without singularities, cannot have more odd circuits than the connectivity-number of the unique odd sheet of the surface. *P. Du Val* (London).

Rosina, B. A. Nuove considerazioni sulla famiglia delle quadriche generalizzate. Ann. Univ. Ferrara. Sez. VII. (N.S.) **5** (1956), 1–10.

A generalised quadric is defined as a surface of order $2n$ whose complete intersection with the plane at infinity is a conic counted n times. A diametral plane (polar plane of a point at infinity) is parallel to that of the same point at infinity with respect to a quadric meeting the plane at infinity in the same conic. The properties of diametral planes, already well-known for quadrics, are developed at great length from this fact. *P. Du Val* (London).

Gallarati, Dionisio. Una proprietà caratteristica delle rigate algebriche. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **21** (1956), 55–56.

It is familiar that the class of a ruled surface (number of tangent hyperplanes in a general pencil) is equal to its order. It is shown here, by simple manipulations of the arithmetic invariants, that every algebraic surface, with only ordinary singularities (by which seems to be meant only improper singularities), whose class is equal to its order, is ruled. *P. Du Val* (London).

Terracini, Alessandro. Nuove superficie particolari dello S_5 in relazione con le loro linee principali. Univ. e Politec. Torino. Rend. Sem. Mat. **15** (1955–56), 255–266.

Sopra una superficie di S_5, distinta dalla superficie di Veronese e non sviluppabile, esistono notoriamente 5 sistemi semplicemente infiniti Σ_i di linee principali [cf. C. Segre, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) **30** (1921); 200–203, 227–231]. Nella Nota presente l'Autore determina le superficie F di S_5 per le quali due dei cinque sistemi Σ_i sono costituiti di linee piane e gli altri tre sono sistemi „conici", e cioè ciascuno costituito di ∞^1 linee $\mathscr{L}$ situate in altrettanti S_3 passanti per uno stesso piano (sicchè i piani tangenti ad F nei punti di una delle linee $\mathscr{L}$ stanno in uno stesso S_4).

Il risultato è che a meno di omografie esistono in S_5 soltanto due superficie F presentanti la suddetta particolarità; entrambe sono superficie razionali dell'ottavo ordine con sezioni iperpiane ellittiche. Le linee principali piane sono coniche mentre le linee $\mathscr{L}$ sono quartiche razionali (di S_3); ed il fatto che le due superficie siano proiettivamente distinte risulta dal diverso comportamento delle linee $\mathscr{L}$. *D. Gallarati* (Genova).

Burniat, Pol. Superficie algebriche di genere lineare grande. Boll. Un. Mat. Ital. (3) **11** (1956), 504–509.

L'A. donne ici un exemple d'une surface algébrique dont le système canonique pur est irréductible et dont le genre linéaire $p^{(1)}$ a la valeur $8p_a+9$, très proche de la valeur maximale probable $8p_a+10$ donnée par G. Zappa [Rend. Mat. e Appl. (5) **14** (1955), 455–464; MR **16**, 1146]. Il s'agit d'un cône algébrique double $2F$, où F est un cône de l'espace ordinaire, d'ordre n et genre $\omega \geqq 1$, doué seulement de génératrices doubles ordinaires. On prend un cône φ d'ordre $2n-4$ biadjoint à F et tangent à F le long de $n+2\omega-2$ génératrices ne formant pas un groupe de la série linéaire découpé sur F par les cônes adjoints d'ordre $n-2$; on prend encore un groupe ψ de $2\pi+2$ plans d'un faisceau où $\pi \geqq 1$; et l'on considère F comme un cône double dont la courbe de diramation est l'intersection de F avec la surface $\varphi+\psi$. Le calcul des caractères de $2F$ donne: $p_g=\pi(\omega-1)$; $p_a=p_g-\omega$; $p^{(1)}=8(\pi-1)(\omega-1)+1$; d'où en effet $p^{(1)}=8p_a+9$. Le genre géométrique p_g et l'irrégularité ω peuvent avoir des valeurs quelconques. *E. Togliatti* (Genova).

Snapper, E. Higher-dimensional field theory. I. The integral closure of a module. Compositio Math. **13** (1956), 1–15.

[Parts II and III were reviewed in MR **18**, 512] Soit E une extension de type fini d'un corps F; on appelle module de E tout sous espace vectoriel (sur F) de dimension finie de E. Sommes et produits de modules sont des modules. On appelle fermeture intégrale d'un module M de E l'ensemble des éléments x de E pour lesquels il existe un module $L \neq (0)$ tel que $xLCML$; c'est un module de E, noté $i(M)$, et aussi l'ensemble des x de E qui satisfont à des équations de la forme

$$x^n+a_1x^{n-1}+\cdots+a_n=0,$$

où $a_j \in M^j$. Par adjonction d'une indéterminée à E on montre que, si M est un module de E, il existe un entier k tel que, pour tout $n \geqq k$ et tout $s \geqq 0$, on a $i(M^{n+s})=i(M^n)\cdot M^s$. Les dimensions sur F des modules M^s, $i(M^s)$ et $M'M^s$ (M': autre module) sont des polynômes en s

pour s grand; leur degré est le degré de transcendance de $F(a^{-1}M)$ sur F pour $a\neq 0$ dans M. Plus généralement, si $M_1, \cdots, M_q$ sont q modules donnés, la dimension sur F de $i(M_1^{n(1)}\cdots M_q^{n(q)})$ est un polynôme en $n(1), \cdots, n(q)$ lorsque ces entiers sont assez grands.

P. Samuel (Clermont-Ferrand).

Gallarati, Dionisio. Sopra una particolare classe di varietà, ed una proprietà caratteristica delle V_r razionali rappresentabili sul sistema lineare di tutte le quadriche di S_r. Univ. e Politec. Torino. Rend. Sem. Mat. **15** (1955–56), 267–280.

Le V_r di S_ρ cogli S_r tangenti a due a due incidenti si ottengon tutte come trasformate omografiche della $V_r^{2^r}$ razionale di Veronese, se $r\geqq 2$ e $\rho\geqq r(r+3)/2$; fanno eccezione i casi in cui quegli S_r tangenti seghino uno spazio fisso S_b in ispazi S_a con $a\geqq b/2$. *G. Scorza-Dragoni.*

Lang, Serge. Unramified class field theory over function fields in several variables. Ann. of Math. (2) **64** (1956), 285–325.

Let K be a field of algebraic functions of several variables over a finite field k, and let L/K be a finite Abelian extension of K. Let $\mathfrak{o}$ be a regular locality in K (i.e. the local ring in K of a simple point of a model of the field K/k), and let $\mathfrak{p}$ be its maximal ideal. If $\mathfrak{o}$ is unramified in L, then we may attach in the usual manner to $\mathfrak{o}$ (or to $\mathfrak{p}$) a Frobenius symbol $(\mathfrak{p}, L/K)$ which is an element of the Galois group G of L/K; this mapping may be extend to a homomorphism φ (the reciprocity map) into G of the free group $\mathfrak{Z}$ generated by all regular localities of K which are unramified in L. The study of the reciprocity map is the main object of class-field theory; the main problems are to determine its image and its kernel. From analogy with class field theory, we expect the image of φ to be the whole of G. As for its kernel, there is a group which is trivially contained in it: let $\mathfrak{o}$ be regular and unramified in L, and let $\mathfrak{O}$ be a locality of L above $\mathfrak{o}$; then the residue field of $\mathfrak{O}$ is an algebraic extension of degree d of that of $\mathfrak{o}$, and d is the order of $(\mathfrak{p}, L/K)$ if $\mathfrak{p}$ is the maximal prime ideal of $\mathfrak{o}$; the kernel of φ therefore contains $d\mathfrak{o}$. Let us call trace group the group generated by the elements $d\mathfrak{o}$ for all regular localities $\mathfrak{o}$ which are unramified in L. From now on, let us assume that the extension L/K is unramified, which means that every normal locality in K is unramified in L (the author proves that it is sufficient that the condition be satisfied for all localities which belong to some normal complete model of K). Then, from analogy with class-field theory, we expect the kernel of the reciprocity map to contain, besides the trace group, a certain subgroup of $\mathfrak{Z}$ which would be analogous to the group of principal ideals in number theory. In the present paper, the author constructs such a group for a certain type of unramified abelian extensions L/K which will be described presently. From now on let there be given a fixed normal projective model V of K/k, and let us restrict $\mathfrak{Z}$ to be the group generated by the local rings of the simple points of V (this entails no serious loss of generality). Let A be the Albanese variety of V and α a canonical mapping (defined over k) of V into A. Let B be an Abelian variety defined over k and which admits a surjective homomorphism λ of finite kernel onto A, defined over k. Then we may associate to B and λ an unramified extension as follows: the set $\lambda^{-1}(\alpha(V))$ turns out to be a variety C; if U is the set of points $(x, y)\in V\times C$ such that $\alpha(x)=\lambda(y)$ (x simple on V), then the projection $V\times C\to V$ induces a map $U\to V$ which is an unramified covering of the set of simple points of V; if L is the field of functions on U rational over k, then L/K is an Abelian unramified extension. Such extensions, and those which may be obtained from these by combination with extensions of the basic field, are called extensions of the Albanese type. Let $\mathfrak{Z}_0$ be the group of elements of degree 0 in $\mathfrak{Z}$; then α defines a homomorphism of $\mathfrak{Z}_0$ into the group A_k of rational points over k of A; the kernel of this homomorphism is called the Albanese kernel. This is the group which takes the place of the group of principal ideals in number theory. In fact, the author proves the following statements, which constitute a complete generalisation of class-field theory: if L/K is of Albanese type, the reciprocity mapping is surjective and its kernel is generated by the trace group and the Albanese kernel; to every subgroup Γ in $\mathfrak{Z}$ containing the Albanese kernel and of finite index, there is associated a unique extension L/K of the Albanese type such that Γ is the kernel of the reciprocity map of L/K; there is a limitation theorem for every Galois extension of K.

The extensions of the Albanese type are certainly not all the unramified Abelian extensions of K. Let for instance X be a rational divisor of V such that, for some $n>0$, nX is linearly equivalent to 0; set $nX=(\varphi)$, $\varphi\in K$; assume that n is prime to the characteristic of K and that k contains the nth roots of unity. Then $K(\varphi^{1/n})$ will be an unramified Abelian extension of K, but will not be of the Albanese type if X is not algebraically equivalent to 0. However, the author proves a result which implies that, if there is no torsion in the Neron-Severi group, then every Abelian unramified extension of degree prime to the characteristic is of the Albanese type; an exemple due to Serre shows that the condition of the degree being prime to the characteristic cannot be omitted from this statement. *C. Chevalley* (Paris).

Kirby, David. Invarianti topologici d'un insieme di elementi differenziali curvilinei. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **21** (1956), 66–71.

The author considers a set of n elements of order $h\geqq 2$, which have in common an element E of order $h-1$, and which belong to an algebraic variety V_n, not singular at the center 0 of E. To any such set, there is associated a certain number of invariants relative to the group of transformations T which are biregular at the point 0. The smallest number n of such elements which have topological invariants, depend on the dimension of n of V_n and the number s of points that are ultimately coincident at the point 0 of E. If $s=1$, there results a simple extension of a theorem of B. Segre, namely that, in general, $(n+1)$ elements possess $(n-1)$ independent invariants. If $s>1$, two elements possess $(s-1)$ independent invariants. Finally a set of $(n-s+2)$ elements of order h which have in common an element E of order $(h-1)$ possess $(n-1)$ independent invariants.

J. De Cicco (Chicago, Ill.).

Burau, Werner. Algebraisch-geometrische Bemerkungen zur Darstellungstheorie der klassischen Gruppen. Jber. Deutsch. Math. Verein. **59** (1956), Abt. 1, 1–6.

A brief expository lecture describing the algebraic varieties left invariant, but on whose points the groups are fully transitive, when the most general matrix representations of the linear, or orthogonal, or symplectic group is interpreted as a group of projective transformations. *P. Du Val* (London).

★ **Weil, André. On the theory of complex multiplication.** Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 9–22. Science Council of Japan, Tokyo, 1956.

Etant donnés une variété complète non singulière V et un diviseur positif X sur V, notons $C(X)$ l'ensemble des diviseurs positifs X' sur V pour lesquels il existe deux entiers m, $m'>0$ tels que $m'X'$ soit algébriquement équivalent à mX; on dit que $C(X)$ est une polarisation de V s'il existe, dans $C(X)$, au moins un système linéaire complet ample; une variété polarisée est donc une variété munie d'une classe privilégiée de plongements projectifs. Etude des variétés abéliennes polarisées; notion de rang d'une telle variété. Soit K_0 un corps totalement réel de degré n sur Q; on appelle CM-extension de K_0 la donnée d'une extension quadratique totalement imaginaire K de K_0 et de n isomorphismes φ_i de K dans $\bar{Q}$ dont les restrictions à K_0 sont toutes distinctes; une CM-extension de K_0 détermine une classe d'isogénie de variétés abéliennes A de dimension n, telles que $\mathfrak{A}_0(A)$ contienne K et que la restriction à K de la trace de la représentation de $\mathfrak{A}_0(A)$ définie par l'algèbre de Lie de A soit $\sum \varphi_i$. Les variétés abéliennes polarisées A de rang donné, d'anneau d'endomorphismes donné et correspondant à une CM-extension donnée, sont en nombre fini; caractérisation des corps K_0 correspondants; étude des extensions de K_0 engendrées par les points d'ordre fini de A.

P. Samuel.

Barsotti, Iacopo. Factor sets and differentials on abelian varieties. Trans. Amer. Math. Soc. **84** (1957), 85–108.

Soient k un corps, V une variété irréductible sur k, W une sous variété de V, $\mathfrak{o}$ son anneau local et $\mathfrak{m}$ l'idéal maximal de celui-ci. Une k-dérivation D de $k(V)$ est dite régulière en W si $D(\mathfrak{m})\subset\mathfrak{m}$, et nulle en W si $D(\mathfrak{o})\subset\mathfrak{m}$; on dit qu'une différentielle ω de V (c'est à dire un élément de l'espace vectoriel dual du $k(V)$-espace vectoriel des k-dérivations de V) a un pole en W s'il existe, soit une k-dérivation D de V régulière en W et telle que $\omega D \notin \mathfrak{o}$, soit une k-dérivation D de V nulle en W et telle que $\omega D \notin \mathfrak{m}$; démonstration des résultats classiques sur les paramètres uniformisants, compte tenu de l'inséparabilité; étude des différentielles de première et de seconde espèce, et de leur comportement par extension du corps de base et par passage au produit. Une différentielle ω est dite fermée si l'on a $D(\omega D')=D'(\omega D)$ quelles que soient les dérivations permutables D, D' de $k(V)$; toute différentielle exacte est fermée; toute différentielle sur une courbe est fermée. On suppose désormais que k est algébriquement clos. Les différentielles invariantes sur une variété abélienne A^n ne sont autres que les différentielles de première espèce; elles sont fermées et forment un espace vectoriel de dimension n sur k. Si k est de caractéristique 0, les différentielles semi invariantes de A^n (c'est-à-dire les différentielles qui sont linéairement équivalentes à leurs translatées) ne sont autres que les différentielles de seconde espèce; la dimension sur k du quotient de l'espace des différentielles de seconde espèce par celui des différentielles exactes est $2n$. Soient Γ l'espace des systèmes de facteurs de A à valeurs dans une variété vectorielle de dimension 1, Γ_c celui des systèmes de facteurs constants, $\mathfrak{D}_2'$ l'espace des différentielles de seconde espèce sur A qui sont régulières à l'élément neutre, et $\mathfrak{D}_1$ celui des différentielles de première espèce; établissement d'un isomorphisme canonique de Γ/Γ_c sur $\mathfrak{D}_2'/\mathfrak{D}_1$ et étude de celui-ci.

P. Samuel.

★ **Shimura, Goro. On complex multiplications.** Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 23–30. Science Council of Japan, Tokyo, 1956.

Soient K un corps de nombres de degré $2n$, et A une variété abélienne de dimension n dont l'anneau d'endomorphismes $\mathfrak{A}(A)$ contient l'anneau des entiers de K. Au moyen de la théorie de la réduction modulo $\mathfrak{p}$ l'auteur démontre une généralisation de la relation de congruence de Kronecker sur les fonctions elliptiques, et en déduit des renseignements explicites sur les extensions abéliennes de K. Lorsque K est un corps quadratique imaginaire, ceci donne une forme explicite de la loi de réciprocité.

P. Samuel (Clermont-Ferrand).

Gunning, R. C. Indices of rank and of singularity on Abelian varieties. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 167–169.

Soient M^n une variété analytique complexe compacte, $\Gamma=\sum\Gamma^{(p,q)}$ l'algèbre extérieure de ses formes différentielles fermées ($\Gamma^{(p,q)}$ désignant l'espace des formes de type (p, q)), τ l'homomorphisme de De Rham de Γ sur l'algèbre $H(M, C)=\sum_d H^d(M, C)$de cohomologie de M à coéfficients complexes. One pose

$$H_k{}^d=\tau\Big(\sum_{\substack{p+q=d\\ p,q\geqq k}}\Gamma^{(p,q)}\Big)\cap H^d(M, Q).$$

Caractérisations de la dimension de $H_k{}^d$ et de celles ed deux de ses sous espaces dans le cas où M est une variété abélienne.

P. Samuel (Clermont-Ferrand).

Barsotti, Iacopo. Abelian varieties over fields of positive characteristic. Rend. Circ. Mat. Palermo (2) **5** (1956), 145–169.

Ce mémoire commence par des lemmes sur les homomorphismes séparables et purement inséparables des variétés de groupes. Etant donnés une variété V définie sur un corps k de caractéristique $p\neq0$ et un entier $r\geqq0$, on appelle hyperdérivation de hauteur r de V sur k toute application k-linéaire D de $k(V)$ dans lui-même telle que $D(xy)=xDy+yDx$ pour $x\in k(V)$ et $y\in k(V^{p^r})$; cette notion est voisine de la notion de semi-dérivation introduite par Dieudonné [C. R. Acad. Sci. Paris **227** (1948), 1319–1320; MR **10**, 280], et les méthodes utilisées ici sont inspirées par les travaux récents de Dieudonné sur les groupes analytiques; une hyperdérivation D de $k(V)$ est dite s-spéciale si $D(k(V^{p^s}))=0$. Détermination des hyperdérivations invariantes d'un groupe algébrique. Relations entre les homomorphismes purement inséparables d'un groupe algébrique commutatif G et les idéaux de l'anneau des hyperdérivations invariantes de G. Application à l'étude des systèmes de facteurs d'une variété abélienne A à valeurs dans un groupe V de dimension 1 et de genre 0. On en déduit que le groupe de torsion d'une variété abélienne A est réduit à l'élément unité; on déduit aussi la structure des formes différentielles de première espèce sur A.

P. Samuel (Clermont-Ferrand).

Severi, Francesco. Osservazioni sui sistemi gruppali d'equivalenza lineare e d'equivalenza algebrica per le ipersuperficie d'una varietà. Abh. Math. Sem. Univ. Hamburg **20** (1956), 127–135.

The main results are recalled, and a few new ones are added, concerning linear and algebraic equivalence among hypersurfaces of an irreducible non-singular algebraic variety, with particular attention to the anomalous continuous systems of hypersurfaces and their relations to Picard varieties.

B. Segre (Rome).

Godeaux, L. Sulle involuzioni cicliche appartenenti ad una superficie algebrica. Rend. Sem. Mat. Fis. Milano **25** (1953–54), 101–112 (1955).

A brief expository lecture summarising the author's very copious work on this topic.

P. Du Val.

Dedò, Modesto. Analisi e costruzione effettiva dei gruppi continui di trasformazioni di De Jonquières. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) (8) **21** (1956), 71–79.

The author begins by showing that for a continuous group of plane De Jonquières transformations of order n the multiple fundamental point (the centre), O, and $n-1$ of the $2n-2$ simple fundamental points $K_1, \cdots, K_{n-1}$ are fixed for all transformations of the group. Moreover fixing a transformation J of the group he deduces that the fundamental points of J^{-1} are $K_1, \cdots, K_{n-1}$ and the fundamental points of J conjugate to the K_i.

If the points $O, K_1, \cdots, K_{n-1}$ are fixed in the plane then a group is formed by the totality of De Jonquières transformations J of order n (presumably with the identity transformation) with centre O such that $K_1, \cdots, K_{n-1}$ are fundamental for J, J^{-1}, and the fundamental points of J, J^{-1} different from the K_i are the fundamental points of J^{-1}, J conjugate to the K_i. It is asserted that the group G so formed is continuous.

The second half of the paper shows how, by means of a fixed De Jonquières transformation G can be represented as a group of transformations (the analytic representations) of one of several simple types, whose equations are stated, depending on the parity of n and the way in which the K_i are situated. *D. Kirby* (London).

See also: Kelly and Straus, p. 666.

NUMERICAL ANALYSIS

Numerical Methods

★ **Crandall, Stephen H. Engineering analysis; A survey of numerical procedures.** McGraw-Hill Book Co., Inc., New York-Toronto-London, 1956. x+417 pp. $9.50.

This is an interesting combination of a reference book and a textbook in the computational aspects of engineering, or the numerical aspects of mathematical analysis — whichever you prefer to name it. The book stems from lectures at Imperial College (London) and at M.I.T., where the author is an Associate Professor of Mechanical Engineering. Although published as an Engineering Societies Monograph for engineers, the book contains an impressive array of material which numerical analysts consider their own.

The book's extremely logical outline is given by the author (p. ix) as follows:

Lumped-parameter systems		Continuous systems
Chap. 1	← Equilibrium problems →	Chap. 4
Chap. 2	← Eigenvalue problems →	Chap. 5
Chap. 3	← Propagation problems →	Chap. 6.

Every chapter has the same structure.

Chap. 1 deals with finite systems of linear algebraic equations. As in every chapter, the author starts off by carefully deriving about 5 typical computing problems from their sources in mechanics, electricity or hydraulics. He then reminds us of the properties of general linear systems, adds a bit on "ill condition", and relates definite systems to quadratic forms. Then follows a survey of solving systems by compact elimination. The next 22 pages are devoted to single-step and total-step iteration, and to the Gauss-Southwell class of relaxation methods. There is a good deal of advice on desk computation, and an occasional reference to high-speed automatic computers. Results are usually quoted and referenced but not proved. There are 40 exercises in this chapter and about 40 literature citations in footnotes.

In Chap. 2 matrix notation is introduced. The general eigenvalue problems considered is of form $AX=\lambda BX$, where $A=A'$, and $B=B'$ is positive definite. We soon meet Collatz's "enclosure theorem," (with an implicit invitation (p. 111) to mathematicians to determine all B for which it holds), Rayleigh quotients, variational interpretations, and Schwarz quotients. Besides some common direct and iterative methods for computing λ, with δ^2 extrapolation, we are introduced to Lanczos's n step reduction of A to a tridiagonal form [J. Res. Nat. Bur. Standards **45** (1950), 255–282; MR **13**, 163] to relaxation methods, to the Kato inequalities [J. Phys. Soc. Japan **4** (1949), 334–339; MR **12**, 447], and to Jacobi's diagonalization of A (for $B=I$) and extensions where $B\neq I$. The latter is the only automatic machine method considered. There are 89 exercises.

The third chapter deals with initial-value problems for systems of non-linear ordinary differential equations. Various methods of undetermined coefficients (linear systems) are followed by difference methods. The author distinguishes (p. 167) the "truncation error" of a finite-difference equation from the "discretization error" of the solution. Euler's method gets a careful analysis. Other recurrence methods are classified into open, closed, and Kutta types.

The second half of the book takes up partial differential equations according to the same outline. Variational relationships are outlined carefully. Although the author devotes some attention to other processes he clearly prefers to discuss finite-difference methods — the area of this own best known research in numerical analysis. The solution methods in Chaps. 4 and 5 are mainly that of the Southwellian relaxation school. The ideas of Chaps. 1–3 reach their important application here to difference equations.

The author carefully reminds the reader that a difference procedure whose error is $O(h^2)$ as $h\to 0$ may not in fact be any more accurate for a given h that one with an error $O(h)$. The case is clearly stated, and certainly the reader steeped in asymptotics will need the reminder.

In Chap. 6 the 5 introductory examples concern the cooling of a rod, a boundary-layer wake, the unsteady transverse motion of a beam, the transverse motion of a taut string, and the expansion of a gas behind a piston. There is a detailed exposition of characteristics, followed by a treatment of the stability of difference methods for heat equation $\partial^2\psi/\partial x^2=\partial\psi/\partial t$. On p. 381 the author calls such a method "unstable" if for some initial heat distributions the finite difference solution becomes unbounded as $t\to\infty$. Explicit and implicit methods are discussed, as well as a related fourth order equation. Hyperbolic equations get a briefer treatment, mainly devoted to

characteristic networks. This chapter has 64 exercises and over 50 references.

The book is concluded with a one-page bibliography and apparently very adequate name and subject indexes. There are a profusion of excellent line drawings and tables of sample computations.

It is difficult to think of other comparable books. Perhaps the closest are Salvadori's "Numerical methods in engineering" [Prentice-Hall, New York, 1952; MR **16**, 1154] and Purday's "Linear equations in applied mechanics," [Oliver and Boyd, Edinburgh-London, 1954; MR **15**, 959]. One should also recall Southwell's two books on relaxation [Relaxation methods in engineering science···, Oxford, 1940; Relaxation methods in theoretical physics, Oxford, 1946; MR **3**, 152; **8**, 355]. Salvadori and Purday are shorter, less ambitious, and convey less of a feeling for computing. Southwell has the feeling for computing, but his books are much less readable, largely because the mathematics is cluttered with engineering dimensions and concrete symbolism. Crandall will certainly dominate the field.

The mathematical reader will be very favorably impressed by the way the author puts his problems in nondimensional form and uses the standard notation of mathematical analysis. It is a pleasure to find such interesting and important topics of modern numerical analysis as the Kohn-Kato bounds for eigenvalues, the finite iteration of Lanczos, Hestenes, Stiefel and others, and various forms of extrapolation to the limit.

It is disappointing to see so little real contact with the methods and problems of computing with automatic digital computers. For example, one of the major contributions to solving Laplace's equation on an automatic computer, the successive over-relaxation idea of Frankel and Young, is pretty well buried in a short foot-note on page 257, while page after page are dedicated to good pencil-and-paper methods. Our disappointment is made unnecessarily bitter by the hopes raised in the first sentence of the preface: "The advent of high-speed automatic computing machines is making possible the solution of engineering problems of great complexity." In fact the book is written largely about desk computing methods. The reviewer doubts that desk computing methods are being used any more to solve a significant number of the problems of this book, at least in the U.S.A.

Minor criticisms: The author's proof on p. 46 of the convergence of the Gauss-Seidel single-step iterative method for solving linear systems contains Gauss-Seidel single-step iterative method for solving linear systems contains the now classical error of assuming that a decreasing sequence of values of a nonnegative quantity Φ must decrease to 0. It is doubtful that the discretization error in computing eigenvalues of irregularly shaped domains is such that h^2 extrapolation will be applicable, despite the author's remark on p. 323.

The author is plainly very competent in engineering analysis, in mathematics, and — most important of all here — in a feeling for numerical processes. The book is highly recommended for engineers and for mathematicians with any interest in computing. And the author shares the feeling of his colleagues that a reasonable amount of exposure to this sort of material would greatly improve the training of any undergraduate mathematics major.

G. E. Forsythe (Stanford, Calif.).

von Hoerner, Sebastian. Herstellung von Zufallszahlen auf Rechenautomaten. Z. Angew. Math. Phys. **8** (1957), 26–52.

The arithmetical generation of "random" numbers by the methods of residue classes and mid-squaring [see, e.g., Juncosa, Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Rep. No. **855** (1953); MR **15**, 559], are discussed. The author suggests the following method: let x_i be the ith number generated, x_{i+1} is then obtained as follows:

$$x_{i+1}=\begin{cases} x_i^2 \text{ if } x_i^2\leq b, \\ x_i^2+a \text{ if } x_i^2>b, \end{cases}$$

where for all i $b\leq x_i<1$, and where the numbers a and b satisfy the conditions that $a+b<1$, $b^2+a>b$, and $b>\frac{1}{4}$. It follows that $b\leq x_i<b+a$.

The author considers this procedure with respect to questions of periodicity or cycles, stable cycles, and degeneration. The discussion is far from adequate. For example: on page 39 one finds histograms of the frequency of cycles for two particular cases. These histograms are compared to their "theoretical" distributions. Without presenting any statistical tests the author concludes, "in beiden Fällen sehr schön mit der Theorie übereinstimmen". {The reviewer does not feel there is good agreement between the observed results and the idealized theory and also that what the author calls the "theoretical" distribution [the distribution obtained from equation (8)], is in fact a questionable approximation.} The author concludes with a discussion of a method for the physical generation of random numbers utilizing electronic computers. *M. Muller* (Princeton, N.J.).

Jessop, W. Neil. Monte Carlo methods and industrial problems. Appl. Statist. **5** (1956), 158–165.

Two industrial situations where Monte Carlo (Empirical Sampling) techniques were applied are cited. Using accumulated knowledge of equipment life as the underlying distribution of life-length of equipment, a problem in plant equipment replacement and maintenance is discussed. The other example concerned a machine servicing problem. *M. Muller* (Princeton, N.J.).

★ **Gaskill, I. Adjustment of the astrogeodetic triangulation network.** Project MO-011. Army Map Service, Washington 25, D. C., 1956. ii+64 pp.

A specific procedure is set forth for the adjustment of a large network of first-order triangulation in a single block by means of the variation of coordinates method (v.c.m.). By "large" is meant an adjustment of well over 1000 normal equations.

In the v.c.m., preliminary coordinates are first computed, and the subsequent adjustment is performed on the linear differentials of these coordinates. The scheme is outlined both for grid and geographic coordinates. The next step is one which is done only for large networks — a Laplace azimuth adjustment. H. Wolf's method [Veröff. Inst. Erdmessungs **4** (1949), 30–38] is adapted to the v.c.m. and the adjustment is made over the mean errors of the astronomical observations and the mean errors of geodetic transport, both corrections being properly weighted. The resulting adjusted Laplace azimuths form one set of condition equations (c.e.); other c.e. are obtained from at least one established position and the fixed (base-extended) sides.

Before setting up the normal equations (n.e.), the observation equations (o.e.) are serially arranged so that

the non-zero terms of the normal set fall as closely to the diagonal as is possible. After the n.e. are obtained from the o.e. and c.e., Schreiber's rule is applied to reduce the number of unknowns. Let $AX-L=0$ be the reset of n.e. in matrix notation. The physical solution for X is limited by the capacity of the Univac I (where a single tape can handle a maximum of 400 n.e.), to which the solution has been adapted. A must thereby be divided into submatrices A_i and a triangularization procedure is carried out on A_i, resulting in $TX-M=0$, where T is triangular, and T_i (of the same order as A_i) is also triangular. The triangularization of A is the main part of the procedure, after this has been completed, the back solution can be readily accomplished. Various checks, not only on the n.e. solution, but on the consistency of the complete network, are described.

The explanation of the entire procedure is quite detailed and composed in an expository style.

B. Chovitz (Washington, D.C.).

Gotthardt, E. Über die rationellste Lösung im allgemeinen Fall der Ausgleichung. Z. Vermessungswesen **82** (1957) 18–22.

Discursive remarks and operational counts for various methods of solving the least-squares problem with side conditions. *A. S. Householder* (Madison, Wis.).

Marchant, R. Modalités pratiques d'application de la méthode des moindres carrés. Assoc. Actuair. Belges. Bull. no. **58** (1956), 25–37.

This is an account of the various numerical methods available for solving over-determined systems of linear algebraic equations. The discussion is in matrix notation. A more complete account of the subject is promised.

D. H. Lehmer (Berkeley, Calif.).

Makover, S. G. Solution of a system of normal equations with the aid of matrices. Astr. Ž. 33 (1956), 423–439. (Russian)

This is a survey of elementary matrix algebra. followed by a matrix formulation of the method of least squares for inconsistent linear algebraic systems. A known compact elimination scheme is expounded for the numerical solution of the normal equations, followed by the efficient computation of certain weights. {Such methods are described in P. Dwyer, Linear computations, Wiley, New York, 1951; MR **13**, 283 — reviewer's reference.}

The introduction states that at the moment [Soviet] astronomers and geodetists make little use of such compact methods, perhaps because the major Soviet text, N. I. Idel'son [The method of least squares and the theory of mathematical processing of observations, Izdat. Geodez. Kartograf. Lit., Moscow, 1947] does not treat matrices. There is no reference to machine computation.

G. E. Forsythe (Stanford, Calif.).

Krasnosel'skiĭ, M. A. On some methods of approximate calculation of the eigenvalues and eigenvectors of a positive definite matrix. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 3(69), 151–158. (Russian)

An exposition of related methods, mostly known, for calculating the least eigenvalue λ_n of a positive definite symmetric matrix A of order n. All methods apply with simple modifications to getting the largest eigenvalue λ_1, and all of them are definable also for infinite-dimensional spaces.

Each method yields a sequence x_k converging to e_n, the eigenvector belonging to λ_n. In most of the methods x_{k+1} is a scalar multiple of $x_k-c_kAx_k$, where c_k is a scalar. The various methods are unified by geometrical constructions x_{k+1} from x_k.

The methods are due to Kantorovič [Uspehi Mat. Nauk (N.S.) **3** (1948), no. 6(28), 89–185; MR **10**, 380], Kostarčuk [Dokl. Akad. Nauk SSSR (N.S.) **98** (1954), 531–534; MR **16**, 863], Krasnosel'skiĭ and Kreĭn [Mat. Sb. N.S. **31(73)** (1952), 315–334; MR **14**, 692], Lanczos [J. Res. Nat. Bur. Standards **45** (1950), 255–282; MR **13**, 163], and Galërkin.

It is noted that any method can be modified by a change in the definition of the inner product from (x, y) to $(A^p x, y)$. *G. E. Forsythe* (Standford, Calif.).

Scotto Lavina, Giovanni. Sul calcolo ed affinamento delle caratteristiche delle vibrazioni dei sistemi elastici ad n gradi di libertà. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **21(90)** (1956), 89–106.

The iterative procedure here discussed for the calculation of eigenvectors and eigenvalues of a real positive definite symmetric matrix is analogous to the method of Kellogg for integral equations except that a different normalization is adopted for the approximating eigenvectors. Also, in calculating an eigenvector and eigenvalue systematic corrections are introduced et each step to allow for the inevitable inaccuracies in the previous calculations corresponding to the larger eigenvalues.

D. C. Lewis (Baltimore, Md.).

Schwarz, Hans-Rudolf. Ein Verfahren zur Stabilitätsfrage bei Matrizen-Eigenwertproblemen. Z. Angew. Math. Phys. **7** (1956), 473–500.

The paper is concerned with the "stability problem" where for a given real or complex matrix A the number of eigenvalues with positive real part is to be determined. After a brief review of the classical criteria (Routh, Hurwitz) the author turns to the theorem by H. S. Wall [Analytic theory of continued fractions, Van Nostrand, New York, 1948, p. 182, MR **10**, 32] who associates with a given polynomial $P(z)=z^n+c_{n-1}z^{n-1}+\cdots+c_0$ the "alternant" of $P(z)$, i.e. the polynomial

$$Q(z)=\tfrac{1}{2}(P(z)-(-1)^n\overline{P}(-z))=p_{n-1}z^{n-1}+iq_{n-2}z^{n-2}+\cdots \qquad (c_k=p_k+iq_k)$$

and forms the continued "test fraction"

$$\frac{Q(z)}{P(z)}=\frac{1}{|r_1z+s_1+1}+\frac{1}{|r_2z+s_2}+\cdots+\frac{1}{|r_nz+s_n},$$

where the r_k are real, the s_k pure imaginary. If m of the r_k are positive and $n-m$ negative, then m roots of $P(z)$ have negative, and $n-m$ positive real parts. The author derives this criterion by constructing a Sturm chain for the polynomial $P(z)$; this leads to a variant of the criterion which enables him to make the decision by means of certain coefficients R_k, S_k occurring in the formation of the chain. Since these coincide with the r_k, s_k respectively, Wall's theorem is indeed proved again. By an elementary transformation the test fraction becomes

$$\frac{a_1}{|a_1+b_1-z}+\frac{a_2}{|b_2-z}+\cdots+\frac{a_n}{|b_n-z};$$

this suggests taking $P(z)$ as the characteristic polynomial

of the Jacobi matrix

$$\begin{pmatrix} a_1+b_1 & a_2 & 0 & & \\ -1 & b_2 & a_3 & \ddots & \mathbf{0} \\ 0 & -1 & b_3 & \ddots & \\ & & \ddots & \ddots & b_{n-1} \quad a_n \\ & \mathbf{0} & & \ddots & -1 \quad b_n \end{pmatrix}$$

where the a_k are real, the b_k pure imaginary numbers. Then it is seen that the number of positive terms in the sequence $a_1, a_1a_2, a_1a_2a_3, \cdots, a_1a_2\cdots a_n$ gives the number of eigenvalues of N which have positive real parts. Now it remains to reduce the given matrix A by similarity into the Jacobi form N. This reduction is "in general" possible and is carried out first for real matrices A; the method fails in the case of derogatory matrices, but no closer investigation of such cases is made here. The method is extended to complex matrices. The paper concludes with two numerical examples, 4×4-matrices, one real with integral elements, one complex.

H. Schwerdtfeger (Melbourne).

Zeuli, Tino. Miglioramento del metodo d'iterazione per la risoluzione dei sistemi di equazioni non lineari. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **90** (1955–56), 362–368.

In a previous note [same Atti **88** (1954), 259–264; MR **16**, 523] the author gave an iterative method for approximation to the root of an equation in one unknown. In the present note, the author extends the method to systems of simultaneous non-linear equations. The method is worked out in detail for two equations in two unknowns. Let (A) $x=\phi(x, y)$, $y=\psi(x, y)$, and let (X, Y) be a first approximation to the solution of these equations. Let

$$m_1=\phi'_x(X, Y), m_2=\phi'_y(X, Y), n_1=\psi'_x(X, Y),$$

$$n_2=\psi'_y(X, Y), J=\begin{vmatrix} 1-m_1 & -m_2 \\ n_1 & 1-n_2 \end{vmatrix}\neq 0.$$

Write the system (A) in the form

$$x-m_1x-m_2y=\phi(x, y)-m_1x-m_2y\equiv F(x, y),$$

$$y-n_1x-n_2y=\psi(x, y)-n_1x-n_2y\equiv G(x, y).$$

Then the recurrence relations for better approximations to the root of the system are $x=(1-n_2)F/J+m_2G/J$, $y=n_1F/J+(1-m_1)G/J$. *E. Frank* (Chicago, Ill.).

Sagni, Giancarlo. Sul calcolo numerico delle radici n-esime. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. **90** (1956), 255–266.

The author presents an optimal recursive procedure for computing the nth root of a number a. The procedure determines successive approximation $y_0, y_1, y_2, \cdots$ by the iterations of the general form

$$y_{r+1}=y_r\frac{a^2+\alpha a y_r^n+\beta y_r^{2n}}{\gamma a^2+\delta a y_r^n+\varepsilon y_r^{2n}}.$$

The case in which

$$\alpha=\delta=(4n-2)/(n+1),$$

$$\beta=\gamma=(2n-1)(n-1)/[(2n+1)(n+1)],$$

$$\varepsilon=1,$$

gives the most rapid convergence the relative error at each step being of the order of the fifth power of the previous error. The expansion of this relative error is given up to the seventh power of the previous error. A number of examples of this and previous formulas are worked out to illustrate their effectiveness.

D. H. Lehmer (Berkeley, Calif.).

★ **Southwell, R. V. Relaxation methods in theoretical physics. A continuation of the treatise Relaxation methods in engineering science.** Oxford, at the Clarendon Press, 1956. vii + pp. 249–522 (7 plates). $8.80.

This is the third volume by the same author on relaxation methods. The first, "Relaxation methods in engineering science" [Oxford, 1940; MR **3**, 152] dealt primarily with problems of statics, frameworks, continuous beams, electrical networks, and frequencies of vibrations. The second, "Relaxation methods in theoretical physics" [Oxford, 1946; MR **8**, 355] had as its main topic the numerical solution by successive approximations of second order elliptic partial differential equations. The present volume is a continuation of the second volume, even the chapters being numbered in sequence. The titles of the chapters indicate the scope of material covered. Chaps. 7 and 8. Relaxation methods applied to biharmonic problems. I, II. Chap. 9. 'Two-diagram technique' as an alternative to use of the biharmonic pattern. Chap. 10. Other applications of 'two-diagram technique'. Stress systems in solids of revolution. Chap. 11. Eigenvalue problems: 'Normal' vibrations of mechanical and electrodynamical systems. Chap. 12. The elastic stability of flat plating subjected to tractions in the plane of its middle surface. Chap. 13. Non-linear problems. Large flexural distortions of flat elastic plates. Chap. 14. Non-linear problems in hydrodynamics and plasticity. Chap. 15. Further developments: 'three dimensional relaxation; heat conduction and other 'transient' problems. Like its predecessors this volume reflects its author's vast experience in the actual solution of practical problems. No less than sixty-two numerical examples are presented in detail, many having a large number of interior points of the net at which values are to be calculated. Many of the problems also involve irregular boundaries and complicated boundary conditions. The problems are exhibited by detailed diagrams, many on folded sheets bound into the book. An example showing the complexity of some problems is the computation of stresses in the plate frame of a locomotive. Not only is the boundary quite irregular but the distribution of the load is also complicated. Everyone seeking practical know-how in relaxational techniques will welcome this valuable addition to the literature of numerical methods. *W. E. Milne.*

Collatz, L. Approximation von Funktionen bei einer und bei mehreren unabhängigen Veränderlichen. Z. Angew. Math. Mech. **36** (1956), 198–211. (English, French and Russian summaries)

After a concise but comprehensive summary of the approximation problem, the author analyzes situations in which the general theorem of Haar does not apply. For example, in a closed strictly convex region B of the (x, y) plane let $g(x, y)\in C^{(1)}$. Then there is exactly one linear function Q for which $\max|Q-g|$ assumes its smallest value, ρ. Examples show that the strong convexity and differentiability are both necessary.

To determine whether a given Tschebyscheff-type approximation can be improved (and if so how much) the author introduces Assumption V: To the functions u_ν of n variables in a region B let there correspond two point sets M_1 and M_2 with the following property: There is no linear combination $Q=\sum a_\nu u_\nu$ which is positive in all

points M_1 but negative in all points M_2. By examples the author shows how this assumption can be verified in important practical cases. Its relevance to the approximation problem is shown by the following: For a given linear combination Q let the error $\varepsilon=Q-g$ be positive in all points of one of the two sets M_1 and M_2 and negative in all points of the other. Let $m=\min|\varepsilon|$ on M_1 and M_2 while $m'=\max|\varepsilon|$ on the whole region B. If V holds then $m\leqq\rho\leqq m'$. The utility of this criterion is demonstrated by examples in which m and m' are so close that a significantly better approximation can be obtained only by taking another term in Q.

Since solutions of elliptic and parabolic differential equations satisfy a maximum principle, a Tschebyscheff-type approximation on the boundary leads to the same approximation for the solution. Typical application: In a problem on heat conduction a certain two-term approximation

$$Q=a_1e^{-t}\sin x+a_2e^{-9t}\sin 3x$$

for the solution u gives $0.00586\leqq\rho\leqq0.00592$ on the boundary. Hence $|Q-u|\leqq0.00592$ throughout; and no choice of a_1 and a_2 would give an approximation better than $|Q-u|\leqq0.00586$. *R. M. Redheffer* (Vienna).

Hersch, Joseph. Le produit de composition des opérateurs aux différences; application aux vibrations de plaques encastrées. C. R. Acad. Sci. Paris **244** (1957), 299–302.

To find (or rather to approximate) numerical solutions of a given differential equation, the equation may be replaced by a difference-equation. The author shows imperfections of the classical method, for instance it cannot be adapted to the conditions at the limits. He indicates methods to determine equations that are exactly satisfied at the points of division. Moreover these methods may be adapted to the conditions at the limits.

When the first member of the differential equation may be written as the product of two differential operators with constant coefficients and the difference-equivalent of each of them is known, the law of composition to find the equivalent of the original differential equation is given.

As an example the equation of vibrating plates is considered. It leads ultimately to a transcendental equation. As a numerical example the case of a hexagonal plate clamped along one side is treated. *H. Bremekamp.*

See also: Maximon, p. 650; Bernstein and Truesdell, p. 654; Miller, p. 658; Brown, p. 678; Nordbotten, p. 683; Meksyn, p. 693; Danø, p. 708.

Tables

Ray, W. D. Sequential analysis applied to certain experimental designs in the analysis of variance. Biometrika **43** (1956), 388–403.

The purpose of the paper is to provide tables for a sequential test of the general linear hypothesis for an experiment designed in a completely randomized design or in a randomized complete block design. Type I and Type II errors of size $\alpha=\beta=.05$ and treatment numbers ranging from 2 to 6, 7, 8, or 10 are considered in constructing the tables. Likewise, values of $\delta=0.5$, 1.0, and 2.0 are used where $\delta=$sum of squares of postulated true treatment effects divided by the product of the number of treatments and the true error variance of a single observation. Conjectural approximations to the expected sample sizes of the sequential processes considered, are also given. The apparent saving in sample size of the sequential procedure is of the order of one-third to one-half of the fixed sample size. *W. T. Federer.*

See also: Macdonald and Brachman, p. 652; Parker, p. 657.

Machines and Modelling

Astrahan, M. M.; Housman, B.; Jacobs, J. F.; Mayer, R. P.; and Thomas, W. H. Logical design of the digital computer for the SAGE System. IBM J. Res. Develop. **1** (1957), 76–83.

This paper describes what is probably the largest, fastest, and most expensive automatic, stored program digital computer in operation, at least in the United States. Designed for application to the SAGE Air defense system, it is actually a general-purpose digital computer with specialized features. Its high-speed magnetic core storage contains 8,192 words, each of 32-bits plus parity digit. Approximately 100,000 words of auxiliary drum storage and also provided, with five magnetic tape units and a cathode ray tube display system.

The computer is almost completely duplexed for the purpose of reliability. Its multiplication speed for 16-bit numbers is 17.5 microseconds, probably the limit for the class of circuitry involved. It also contains a standard indexing or "*B*-Line" system; unfortunately the authors have not credited Williams or Turing of Manchester, or the designers of the MIDAC and Datatron in the U.S.A., who had priority with this technique. Input-output allows simultaneous operation and data-handling through buffer-storage drums.

To the mathematician, the following are important: 1) This is the first computer to perform multi-dimensional operations (arithmetic on two-dimensional vectors); 2) the scheduling problem for computers employing such concurrent operation is acute; much further analysis will be needed; 3) this computer, of which many copies will be made, has not been planned so that upon its miltary obsolescence, it could be of use to university or other research organization in dire need of computation facilities; and 4) the extremely complex logical problem required in the "programming" of problems on such equipment is well worth sincere mathematical investigation. *John W. Carr, III* (Ann Arbor, Mich.).

Toma, V. Le système arithmétique d'une machine électronique à calculer, en cours d'étude et de réalisation à l'Institut de Physique de l'Académie de la République Populaire Roumaine. Acad. R. P. Romîne. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. **8** (1956), 223–235. (Romanian. Russian and French summaries)

★ **Brown, J. A. C. An experiment in demand analysis. The computation of a diet problem on the Manchester computer.** Conference on linear programming, May, 1954, pp. 41–53; discussion, 54–55. Ferranti Ltd., London.

A linear programming approach. *H. Wold.*

Bendat, Julius S. Mathematical analysis and analog simulation of atmospheric turbulence gust velocities. J. Aero. Sci. **24** (1957), 69–70.

This paper describes apparatus for generating an approximation to observed power spectra of atmospheric turbulence gust velocities by analogue means. A "white noise" (constant power spectral density) signal passed through a linear system with a transfer function $Y(p)$ produces an output signal with a power spectral density function $G_0(\omega)=N|Y(j\omega)|^2$, where N is the white noise constant. The author suggests that a transfer function

$$Y(p)=[A_1A_2(p+b)]/[p^3+(a+b)p+(ab+A_1A_2A_3)],$$

which is readily achieved by means of a simple feed-back circuit, may be used to give adequate representation of observed power spectra of atmospheric turbulence gust velocities. *J. G. L. Michel* (Teddington).

Sabliet, Samuel. Introduction à une théorie des ensembles basée sur les nombres premiers assimilable par les calculateurs électroniques. C. R. Acad. Sci. Paris **244** (1957), 35–38.

The author proposes to do formal mathematics on a digital computer by assigning different prime numbers to different functions thereby assuring uniqueness of representation for combinations of functions. However, this method fails whenever such combinations include sums. *P. Rabinowitz* (Rehovoth).

See also: Leech, p. 642.

PROBABILITY

★ **Novák, Josef. Die topologische Struktur der Wahrscheinlichkeitsfelder.** Bericht über die Tagung Wahrscheinlichkeitsrechnung und mathematische Statistik in Berlin, Oktober, 1954, pp. 17–21. Deutscher Verlag der Wissenschaften, Berlin, 1956.

Let $\varphi(C)$ be a function defined for all sets C that belong to $\mathfrak{C}$, a set of sets. The function is said to be continuous at the element C if $\varphi(C_n)\to\varphi(C)$ whenever $C_n\to C$ in the sense of the theory of sets. The function is continuous on $\mathfrak{C}$ if it is continuous at each element C. The paper consists of statements and partial proofs of a number of results of which the following one is typical. A measure function that is defined to be finite and real for every set belonging to a ring of sets is continuous if and only if it is bounded. Hence every probability function is continuous, a probability function being defined in effect as by Kolmogoroff [Grundbegriffe der Wahrscheinlichkeitsrechnung, Springer, Berlin, 1933; see also Novák and Novotný, Czechoslovak Math. J. **3(78)** (1953), 291–296; MR **15**, 691]. *I. J. Good* (Cheltenham).

★ **Rényi, Alfred. Axiomatischer Aufbau der Wahrscheinlichkeitsrechnung.** Bericht über die Tagung Wahrscheinlichkeitsrechnung und mathematische Statistik in Berlin, Oktober, 1954, pp. 7–15. Deutscher Verlag der Wissenschaften, Berlin, 1956.

A shortened version of the author's paper in Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **4** (1954), 369–427 [MR **16**, 599]. The axioms are equivalent to those used by Good [Probability and the weighting of evidence, Griffin, London, 1950, pp. 19, 23; MR **12**, 837], but are expressed in terms of sets instead of in terms of propositions. As pointed out by the reviewer, loc. cit. p. 30, the conditional and unconditional axioms are equivalent so long as almost impossible propositions are not allowed to occur as "given". Rényi, however, explicitly allows almost impossible propositions to be given, so as to be able to cope with non-normalizable (possibly infinite) probabilities, as used by physicists. Infinite probabilities are also discussed by Jeffreys [Theory of probability, Oxford, 1939, pp. 21, 114; MR **1**, 151] and by the reviewer [loc. cit., pp. 21, 55–56]; also Popper [British J. Philos. Sci. **6** (1955), 51–57; MR **17**, 378] allows almost impossible propositions to be given. *I. J. Good.*

Loeffel, H. Beiträge zur Theorie der charakteristischen Funktionen stochastischer Verteilungen. Mitt. Verein. Schweiz. Versich.-Math. **56** (1956), 337–384.

An elaboration of the author's previous notes [C. R. Acad. Sci. Paris **240** (1955), 1964–1966, 2293–2294; MR **16**, 937, 911] with some additions. Sample result: If F is a distribution function and $p\in(0,1)$, then $\int_0^x F(x/\alpha)d\alpha^p$ is unimodal with vertex at $x=0$. *H. P. McKean, Jr.*

Simaika, J. Sur une mesure de la dispersion d'une distribution de directions. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 753–756.

Consider a distribution function F on the circle and write $\mathfrak{M}(\cdot)=\int(\cdot)F(d\theta)$. The author has shown [C. R. Acad. Sci. Paris **241** (1955), 1375–1377; MR **17**, 500] that the angle $\alpha=\arg\mathfrak{M}(e^{i\theta})$ that minimizes

$$\Delta(\psi)=(\mathfrak{M}|\,e^{i\psi}-e^{i\theta}|^2)^{1/2}$$

has several advantages over the usual expectation $\hat{\theta}=\mathfrak{M}(\theta)$. Here, he suggests the angle

$$\beta=\cos^{-1}|\mathfrak{M}(e^{i\theta})=2\sin^{-1}(\Delta(\alpha)/2)$$

as a measure of the spread of F; unlike the standard deviation $(\mathfrak{M}(\theta-\hat{\theta})^2)^{1/2}$, β is insensitive to rotations. *H. P. McKean, Jr.* (Princeton, N.J.).

Fréchet, Maurice. Sur la distance de deux lois de probabilité. C. R. Acad. Sci. Paris **244** (1957), 689–692.

X et Y sont deux variables aléatoires de moyennes $\overline{X}$, $\overline{Y}$, d'écarts types σ_X, σ_Y, dont les lois individuelles L, L' sont données. Une définition adéquate de la distance de L et L' est

$$(L,L')=\sqrt{[(\overline{X}-\overline{Y})^2+\sigma_X{}^2+\sigma_Y{}^2-2\rho\sigma_X\sigma_Y]},$$

où ρ est la borne supérieure du coefficient de corrélation entre X et Y, pour l'ensemble des fonctions de répartition $H(x,y)$ à deux dimensions compatibles avec L et L'. ρ correspond à la fonction de répartition

$$H_1=\min[F(x),G(y)],$$

où $F(x)$ et $G(y)$ sont les fonctions de répartition respectives de X et Y.

Une autre définition intéressante de la distance (L,L') est la „distance globale" (au sens de P. Lévy) de X et Y relativement à H_1. *J. Bass* (Paris).

Blum, J. R. On a characterization of the normal distribution. Skand. Aktuarietidskr. **39** (1956), 59–62.

The main result proved is as follows: if $\varphi(t)$ and $\psi(t)$ are characteristic functions, n an integer greater than 1, m_1 and m_2 positive integers each not greater than n, c a real number, and

$$\varphi(t)\psi(t)=e^{ict}\varphi^{m_1}(u)\varphi^{n-m_1}(-u)\psi^{n-m_2}(-u)\psi^{m_2}(u)$$

with $u=t/\sqrt{n}$, then $\varphi(t)$ and $\psi(t)$ correspond to normal distributions.

It is used to give a rapid proof of a theorem of S. Bernstein [Trudy Leningrad, Politehn. Inst. (1941), 3–22] supplying the usual characterization of the normal distribution and also to generalize an alternate characterization due to G. Pólya [Math. Z. **18** (1923), 96–108].

J. Riordan (New York, N.Y.).

Kloss, B. M. Limiting distributions of sums of independent random variables taking values from a bicompact group. Dokl. Akad. Nauk SSSR (N.S.) **109** 1956), 453–455. (Russian)

Employing machinery devised by the reviewer [Proc. Amer. Math. Soc. **5** (1954), 923–929; MR **16**, 796] the author discusses convergence of sequences μ^n, $n=1, 2, \cdots$, where μ is a normalized regular positive measure on a compact group G. The problems and results (the latter being stated without proof) are also expressed probabilistically, in terms of partial "sums" of identically distributed G-valued random variables. In this terminology the central result is the obvious one, that the only stable laws are uniform distributions over compact subgroups of G. [Cf. also Prékopa, Rényi and Urbanik, Acta Math. Acad. Sci. Hungar. **7** (1956), 11–16, and pertinent comments by the reviewer thereof, MR **18**, 25.]

J. G. Wendel (Ann Arbor, Mich.).

Matsuyama, Noboru; and Takahashi, Shigeru. The law of the iterated logarithm for dependent random variables. Sci. Rep. Kanazawa Univ. **4** (1956), 177–182.

Let $\xi_1, \xi_2, \cdots$ be a sequence of m-dependent random variables (i.e., $(\xi_1, \cdots, \xi_r)$, $(\xi_s, \cdots, \xi_n)$ are independent if $s-r>m$) with zero means. Put $S_n=\xi_1+\cdots+\xi_n$, $B_n=E(S_n^2)$. It is shown that if $\int_{|\xi_i|>\varepsilon B_n} \xi_i^2 dP=O((\log n)^{-1-\varepsilon})$ uniformly in i for some $\varepsilon>0$ as $n\to\infty$, and $\liminf_{n\to\infty} n^{-1}B_n>0$, then

$$\limsup_{n\to\infty} S_n(2B_n \log\log B_n)^{-\frac{1}{2}}=1$$

with probability one. *W. Hoeffding.*

Rivkind, Ya. I. Limit theorem of probability theory on compact topological groups. Grodnenskiĭ Gos. Ped. Inst. Uč. Zap. **1** (1955), 51–58. (Russian)

It is well-known that, if n independent random real variables all have the same absolutely continuous distribution, then as $n\to\infty$ the distribution of the fractional part of their sum tends to the uniform distribution over $[0, 1)$. The author obtains a result of this type in a much more general setting. Let Ω be a connected compact topological group. Let $\mu(E)$ be the normalized (two-sidedly) invariant measure on Ω and $P(E)$ be a probability measure on Ω. Let $x_1, x_2, \cdots$ be independent randomly variable elements in Ω all having the distribution determined by $P(E)$. The distribution of the product $x_1x_2\cdots x_n$ is determined by $P^{(n)}(E)$, where

$$P^{(k)}(E)=\int_\Omega P^{(k-1)}(Ex^{-1})P(dE_x) \quad (k=1, 2, \cdots)$$

and $P^{(1)}(E)=P(E)$. Supposing $P(E)$ to be of the form $\int_E p(x)\mu(dE_x)$, with $p(x)$ in the Lebesgue class L_Ω^2, the author shows that $P^{(n)}(E)\to\mu(E)$ as $n\to\infty$. His proof employs a generalized characteristic function.

H. P. Mulholland (Birmingham).

Blum, J. R.; and Rosenblatt, Murray. A class of stationary processes and a central limit theorem. Duke Math. J. **24** (1957), 73–78.

Let $\eta=\{\eta_j, -\infty<j<\infty\}$ be a sequence of mutually independent random variables with a common distribution, and let T be the translation (shift) operator acting on η. Then, if g is a random variable which is a measurable function of η in the usual sense, and if $E\{|g|^2\}<\infty$, $X_n=g(T^n\eta)$ defines a stationary stochastic process. The authors prove that the spectral distribution of this process is absolutely continuous. [Essentially the same result was proved by the reviewer and Leibler, Amer. J. Math. **65** (1943), 263–272; MR **4**, 219.]] Under further moment conditions, it is shown that $\sum_1^N X_i/N^{\frac{1}{2}}$ is asymptotically normally distributed, with mean 0 and variance $2\pi f(0)$, where f is the spectral density of the X_n process. *J. L. Doob* (Geneva).

Ramakrishnan, A.; et Srinivasan, S. K. Sur les intégrales stochastiques associées aux processus ponctuels. Publ. Inst. Statist. Univ. Paris **5** (1956), 95–106.

Les auteurs considèrent d'abord la variable aléatoire

(1) $Y_m(t)=$

$$\int_0^t \phi_m(t_m)dt_m \int_0^{t_m} \phi_{m-1}(t_{m-1})dt_{m-1}\cdots\int_0^{t_2}\phi_1(t_1)n(t_1)dt_1,$$

où les $\phi_j(t)$ sont des fonctions certaines, et où $n(t)$ est une fonction aléatoire discontinue donnée (nombre de points aléatoires situés sur $(0, t)$). Il s'agit de calculer la densité de probabilité de $Y_m(t)$, ou tout au moins ses moments.

Dans la première partie, ils choisissent pour $n(t)$ un processus de Furry: la probabilité pour qu'un évènement aléatoire arrive entre τ et $\tau+d\tau$ est $[1+n(\tau)]\lambda d\tau$. La probabilité pour que $n(t)=n$ est égale à

$$e^{-\lambda t}(1-e^{-\lambda t})^n.$$

Sur une épreuve, on a d'ailleurs $n(\tau)=\sum_{j=1}^{\infty} H(\tau-t_j)$, H fonction d'Heaviside.

Une généralisation de (1) est alors introduite sous la forme

(2) $Y_m(t)=$

$$\int_0^t \phi_m(t_m)dt_m\int_0^{t_m}\phi_{m-1}(t_{m-1})dt_{m-1}\cdots\int_0^{t_1}\varphi_0(t_0)\frac{dn(t_0)}{dt_0}dt_0,$$

La densité de probabilité de Y_m et n satisfait à une équation fonctionnelle qu'on résout par l'intermédiaire d'une fonction génératrice et de transformées de Laplace. Le calcul des moments de $Y_m(t)$ est ensuite repris par la méthode des densités-produits [cf. Ramakrishnan, Proc. Cambridge Philos. Soc. **49** (1953), 473–485; MR **14**, 1100].

Dans la deuxième partie, on élargit la définition de $n(t)$ en supposant que la probabilité pour qu'il existe un point aléatoire entre t et $t+dt$ est $\lambda\psi(n)dt$. La densité de probabilité de n est alors de la forme

$$\psi(n-1)\psi(n-2)\cdots\psi(0)\sum_{k=0}^{n} A_k^n e^{-\lambda\psi(k)t},$$

$$A_k^n=\prod_{j\neq k}\{\psi(j)-\psi(k)\}^{-1}.$$

La méthode des équations aux dérivées partielles utilisée dans la première partie se généralise, et conduit au calcul explicite de la densité de probabilité de $Y_m(t)$ lorsque $m=0$, $\phi_0(t)=1$, et lorsque les ψ_m sont en progression arithmétique.

On examine le cas particulier où la probabilité conditionnelle pour qu'il ne se trouve aucun point entre t_1 et

$t_2>t_1$, étant donné qu'il y a point un aléatoire en t_1, est $\phi(t_2-t_1)$, indépendamment de la distribution des points pour $t<t_1$. A la fonction $n(t)$ ainsi définie, on peut d'ailleurs associer le processus stochastique $x(t)$ représenté par

$$x(t)=a_i,\ t_j\leqq t<t_{j+1}\ (i=0,\pm1,\pm2,\cdots),$$

où les a_i sont des variables aléatoires indépendantes de même distribution, et où les $t_{i+1}-t_i$ sont des variables aléatoires indépendantes de même distribution.

L'étude générale de

$$y_m(t)=\int_0^t \frac{dx(\tau)}{d\tau}\,F(t,\tau)d\tau$$

semble difficile. Il est cependant possible de calculer la densité de probabilité de $\int_0^t x(\tau)d\tau$. *J. Bass* (Paris).

Bellman, Richard. On a generalization of the fundamental identity of Wald. Proc. Cambridge Philos. Soc. **53** (1957), 258–260.

Let $\{z_t, t\geqq 0\}$ be a Markov process with stationary transition probabilities, and let

$$\Phi_N(x,t)=E\{\exp\,(t\sum_1^N z_t)|z_0=x\}.$$

The author states that, under various assumptions on the specified transition probabilities, $\Phi_N(x,t)\sim\lambda(t)^m\psi(x,t)$. If n is a random stopping time in the usual sense, it is suggested that the Wald identity (in which the z_t's are mutually independent) can be replaced by

$$E\{(\exp\,t\sum_1^N z_t)\psi(z_n,t)/[\lambda(t)^n\psi(x,t)]|z_0=x\}=1.$$

It is indicated how the exponential here can be replaced by successive products, and how the the random variables can be replaced by matrix random variables. Proofs and exact hypotheses are deferred. *J. L. Doob.*

Palmer, D. S. Corrigenda to the paper "Properties of random functions". Proc. Cambridge Philos. Soc. **53** (1957), 266.

The original article reviewed in MR **17**, 241.

Miller, Irwin; and Freund, John E. Expected arc length of a Gaussian process on a finite interval. J. Roy. Statist. Soc. Ser. B. **18** (1956), 257–258.

The authors consider a Gaussian stationary process over a finite time interval. They determine the expected value of the arc length of the process and express it in terms of a confluent hypergeometric function.

U. Grenander (Stockholm).

Jackson, R. R. P.; and Nickols, D. G. Some equilibrium results for the queueing process $E_k/M/1$. J. Roy. Statist. Soc. Ser. B. **18** (1956), 275–279.

A queueing process is studied in which the inter-arrival time of customers is proportional to a χ^2 variate, the service time distribution is negative exponential and a single server serves customers in order of arrival. The steady state solutions for (i) the probability that there are n customers in the queue, (ii) the probability that a new customer finds n others in front of him, and (iii) the probability distribution of waiting-time, are found. The limiting case of regular arrivals is discussed.

D. V. Lindley (Cambridge, England).

Derman, Cyrus. A note on nonrecurrent random walks. Proc. Amer. Math. Soc. **7** (1956), 762–765.

Let $X_1, X_2, \cdots$ be independent, identically distributed random variables with a density function $f(x)$ and $EX_i=\mu>0$. Let $S_n=X_1+\cdots+X_n$, $H(x)=\sum_{n=1}^{\infty}P(S_n\leq x)$ and $h(x)=H'(x)$ p.p. According to Chung and Derman [Pacific J. Math. **6** (1956), 441–447; MR **18**, 424], (i) $\liminf_{x\to\infty} h(x)>0$ and (ii) $\limsup_{x\to\infty} h(x)<\infty$ imply that if A is a Borel set of positive real numbers, then (1) $P(S_n\in A \text{ i.o.})=0$ when meas $A<\infty$, (2) $P(S_n\in A \text{ i.o.})=1$ when meas $A=\infty$, where i.o. means infinitely often. The author shows that (i) alone implies (1), while a sufficient condition for (i), if $\mu<\infty$, is either that (ii) hold or that $f(x)\geqq\text{const}>0$ on some x-interval. The proofs depend on the relation $H(x-a)-H(x-b)\to(b-a)/\mu$ as $x\to\infty$ [Blackwell, 3 (1953), 315–320; MR **14**, 994] and an ergodic theorem [Harris and Robbins, Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 860–864; MR **15**, 140]. *P. Hartman.*

Philipson, Carl. A tentative application of the collective risk theory to crop insurance. Skand. Aktuarietidskr. **38** (1955), 201–253.

A diffuse study in ten chapters of stochastic processes which are functions of space and time and of their application (without detailed numerical results) in the insurance of a cultivator's crops. The paper is full of new ideas on the application of existing theory but no new mathematical theorems are involved. *H. L. Seal.*

Philipson, Carl. A note on different models of stochastic processes dealt with in the collective theory of risk. Skand. Aktuarietidskr. **39** (1956), 26–37.

The author discusses the risk process treated by Ammeter [Skand. Aktuarietidskr. **31** (1948), 171–198; MR **10**, 463] and points out some passages which he considers to be not quite clear. He also discusses the process proposed in the last section of one of Arfwedson's papers [ibid. **38** (1955), 37–100; MR **17**, 638] which uses the assumption that the probability of more than one claim during the period t to $t+dt$ is of the order of dt. The relation between this process and the Polya process is discussed. *E. Lukacs* (Washington, D.C.).

See also: Bharucha-Reid, p. 646; Fréchet, p. 646; Mikhail, p. 648; Dugué, p. 650; von Hoerner, p. 675; Ray, p. 678; Schmetterer, p. 681; Vogel, p. 682; Wilson, p. 686; Rogers, p. 686; Rogers and Wilson, p. 687; Rogers, Stanley and Wilson, p. 687; Bass, p. 702; Burkhardt, p. 707; Luchak, p. 707; Burke, p. 707; Kesten and Runnenburg, p. 708.

STATISTICS

★**Schmetterer, L. Einführung in die mathematische Statistik.** Springer-Verlag, Wien, 1956. xxiii+405 pp. $11.65.

As the author states in his Preface, the lack of a modern text-book on Mathematical Statistics in the German language has made itself felt in an unfortunate way. The present book has grown out of the author's lectures at the University and the Technological Institute of Vienna. It begins with an introductory chapter on Mathematical Probability, well and clearly written though, as the author points out, not designed to give a completely rigorous mathematical treatment. Standard topics such as random variables, characteristic functions, various types of distributions etc. are treated. (A minor observa-

tion concerning this chapter: the theorem on p. 55 which is attributed to the reviewer has been wrongly stated.) In the following chapters the applications of Probability Theory to Statistics are given. As the author states, he is mainly concerned with those classical branches of Mathematical Statistics associated with the names of Fisher, Pearson and, above all, Neyman. Accordingly there are chapters on sampling distributions, confidence regions, statistical tests and estimation, etc. A particularly interesting feature is a chapter of about 50 pages on non-parametric methods. Questions connected with decision theory or game theory have not been included. The treatment is clear and accurate, and takes account of recent developments in statistical theory. However, some readers may regret that the author has restricted himself to a purely theoretical text, with practically no hints or illustrations concerned with the possibilities and pitfalls of the applied field. *H. Cramér* (Stockholm).

Vogel, Walter. Asymptotische Eigenschaften von Maximum-Likelihood Schätzwerten bei einem stochastischen Prozess. Monatsh. Math. **60** (1956), 313–321.

An experimenter carries out a series of experiments, each of which is characterized by a parameter s. The outcome of the experiment has a probability distribution depending upon s and upon an unknown parameter θ. The parameter s is allowed to depend upon the results of the earlier experiment. The author studies the maximum likelihood estimate of θ and shows that it is consistent, asymptotically normally distributed and asymptotically efficient under certain given conditions. *U. Grenander.*

Grundy, P. M. Fiducial distributions and prior distributions: an example in which the former cannot be associated with the latter. J. Roy. Statist. Soc. Ser. B. **18** (1956), 217–221.

For f a non-negative non-constant polynomial, let $p_\theta(x)$ be the n-variate density $\prod[c(\theta)f(x_i)\exp -(x_i-\theta)^2/2]$ $(-\infty<\theta<\infty)$. It is shown that the θ-derivative of $P_\theta\{\sum X_i>\sum x_i\}$, though positive valued, is never of the form $p_\theta(x)\psi(\theta)/\int p_\theta(x)\psi(\theta)d\theta$. *J. Hannan.*

Des Raj. Ratio estimation in sampling with equal and unequal probabilities. J. Indian Soc. Agric. Statist. **6** (1954), 127–138.

In a finite population, let x and y be two variates such that advance information is available on x, in particular the sum X of x for the total population is known, while information on y is to be obtained by sampling. As an estimate for the population total Y of y the "ratio estimate" $X(\bar{y}/\bar{x})$ has been used. The author points out that not only, as noted in the past, this estimate is subject to an unknown bias, but that its variance may be larger than the variance $\sigma^2(\bar{y})$ of a simple sample mean even in the case of perfect correlation. The main contribution of this paper is a proposed modification of the ratio estimate which is unbiased. An estimate of its variance is obtained and applications to unistage, stratified, multi-stage and multi-phase sampling are studied. *Z. W. Birnbaum.*

Hannan, E. J. The asymptotic powers of certain tests based on multiple correlations. J. Roy. Statist. Soc. Ser. B. **18** (1956), 227–233.

The author derives a generalised measure of the relative efficiency of two tests of a hypothesis when both test statistics are asymptotically χ^2 distributed (a particular case being the comparison of two multiple correlations). He applies this to compare tests of the significance of regression when residuals follow a first order autoregression, and of the serial dependence of a vector variate obeying a first order vector autoregression. *P. Whittle* (Wellington).

van Eeden, Constance; and Benard, A. General theorems on Wilcoxon's test for symmetry. Math. Centrum Amsterdam. Statist. Afdeling Rap. **S 209 (VP 10)** (1956), 23 pp.

The authors (a) examine the consistency of the test, (b) combine it with Wilcoxon's two sample test, and (c) generalize it by substituting the rank numbers by general weights. *H. Wold* (Uppsala).

Wabeke, Ir Doraline; and van Eeden, Constance. Guide to Wilcoxon's test. Math. Centrum Amsterdam. Statist. Afdeling. Rap. **S 176(M65)** (1955), i+37 pp. (Dutch)

This and the two following reports give thorough expositions with schemes of calculation, illustrations and tables. *H. Wold* (Uppsala).

van Eeden, Constance; and Wabeke, Ir Doraline. Guide to Wilcoxon's test (continuation). Exact treatment in the case of like observations. Math. Centrum Amsterdam. Statist. Afdeling Rap.. **S 176 (M 65A)** (1955), 10 pp. (Dutch)

Benard, A.; and van Eeden, Constance. Guide to Wilcoxon's test for symmetry. Math. Centrum Amsterdam. Statist. Afdeling Rap. **S 208(M 76)** (1956), i+33 pp. (Dutch)

van Eeden, Constance. Distributionfree two-sample tests and the method of the 2×2-table. Statistica, Neerlandica **10** (1956), 157–162. (Dutch. English summary)

The author shows that the distribution-free two-sample tests of Pitman, Wilcoxon, Terry, and van der Waerden, discussed by van Dantzig and Hemelrijk [Bull. Inst. Intern. Statist. **24** (1954), 2ème livraison, 239–267; MR **16**, 941, are equivalent to the method of the 2×2-table when applied to two independent series of Bernoulli trials. *H. A. David* (Melbourne).

Smid, L. J. On the distribution of the test statistics of Kendall and Wilcoxon's two sample test when ties are present. Statistica, Neerlandica **10** (1956), 205–214. (Dutch summary)

The author considers a general situation covered in special cases by Kendall's test without ties, and Wilcoxon's two-sample test without ties. He obtains a recursion relation for the distribution of Kendall's statistic S for testing the hypothesis that two ways of ranking the objects of a population are mutually independent. A sample of N objects is taken, a pair (A, B) is assigned (i) the score $+1$, if A precedes B in both rankings, or again if B precedes A in both rankings, (ii) the score 0 if there is a tie in either ranking, (iii) the score -1, if A precedes B in one ranking, but B precedes A in the other. Then S is the sum of the scores for the $C(N, 2)$ pairs of objects in the sample. The author considers a generalized scatter diagram, with columns designated by the several ties in the first ranking, and rows by the ties in the second. The objects are assigned separate letters. A "case" occurs when the two rankings are completely known. When only

the number of objects in each cell of the table is known, one is said to have a "scheme." If only the sums by rows and by columns are known, one has a "marginal scheme." The author examines in some detail the recursively computed distributions of S for the marginal scheme (1, 8, 3, 3, 6; 9, 12) and finds the two-tailed critical regions corresponding to $\alpha=1\%$ and 5%. The normally approximated values, with continuity correction 1, are seen to fit very closely to the exact values. For this example involving 293930 grouped items, direct computation would have required 193 separate schemes. *A. A. Bennett.*

Arnáiz, Gonzalo. Sampling inspection. Trabajos Estadist. **7** (1956), 221–235. (Spanish)

In this expository paper the author discusses in a rather heuristic manner the sequential probability ratio test and obtains the usual approximations for the probability of errors of type I and II. This is carried out in detail for sampling inspection by attributes. The second part is concerned with fixed sample size testing of the proportion of defective objects assuming normality. In this part the analysis is incorrect in that the normal distribution is used instead of non-central t. *H. Rubin.*

Kramer, Clyde Young. Extension of multiple range tests to group means with unequal numbers of replications. Biometrics **12** (1956), 307–310.

The problem is to examine the differences $\bar{x}_i-\bar{x}_j$ $(i\neq j;\ i,\ j=1,\ \cdots,\ m)$, where $\bar{x}_i$ is the mean of a sample of n_i independent observations from the distribution $N(\mu_i,\ \sigma)$. It is suggested that a "reasonable" procedure is to declare the difference $|\bar{x}_{[i]}-\bar{x}_{[j]}|$ significant at the α level when it exceeds $s[(n_{[i]}+n_{[j]})/2n_{[i]}n_{[j]}]^{\frac{1}{2}}z_{p,\nu}$, where $[i]$ indicates the ith ranked mean, s is the usual within samples pooled estimate of σ based on ν degrees of freedom and $z_{p,\nu}$ is the upper α point of the Studentized range (i.e., of the ratio w/s, where w is the range of $p=|i-j|+1$ independent $N(0,\ 1)$ variates, νs^2 is a χ^2 variate with ν degrees of freedom, and w and s are statistically independent). *W. S. Connor* (Washington, D.C.).

Gatti, Stefania. Sul massimo di un indice di anormalità. Metron **18** (1956), no. 1–2, 181–188.

Proof of the inequality

$$\frac{1}{n}\sum_{i=1}^{n}(a_i-\bar{a})^2\Big/\Big(\frac{1}{n}\sum_{i=1}^{n}|a_i-\bar{a}|\Big)^2\leq\frac{n}{2},$$

where $\bar{a}=n^{-1}\sum_{i=1}^{n}a_i$. *Z. W. Birnbaum.*

Thompson, H. R. Extensions to missing plot techniques. Biometrics **12** (1956), 241–244.

Formulae are presented for estimating missing observations in a randomized complete block experiment with r blocks and t treatments for the particular case when there is one missing observation in each of n treatments $(n\leq t)$ distributed over p blocks $(p\leq r)$ such that there are n_i in the ith block. The results are illustrated with a numerical example. Except for the application, these results are published elsewhere in the literature [viz. Federer, Thesis, Iowa State Coll., 1948; Res. Bull., Iowa State Coll. no. **380** (1951)]. *W. T. Federer.*

Rao, C. Radhakrishna. A general class of quasifactorial and related designs. Sankhyā **17** (1956), 165–174.

Let $v=p_1p_2\cdots p_n$, and let the n-tuples $(x_1,\ x_2,\ \cdots,\ x_n)$, where $x_i=1,\ 2,\ \cdots,\ p_i$ for $i=1,\ 2,\ \cdots,\ n$, be varieties in a design of blocks, each containing k different varieties, such that: (i) each variety occurs r times; (ii) the two varieties $(x_1,\ x_2,\ \cdots,\ x_n)$ and $(y_1,\ y_2,\ \cdots,\ y_n)$ occur together in $\lambda(c_1,\ c_2,\ \cdots,\ c_n)$ blocks, where $c_i=1$ or 0 according as $x_i=y_i$ or not. Such a design is a quasifactorial design, and the author constructs in this paper examples of such designs for the special case that $n=2$ and there are exactly three associate classes. Two main techniques are used, one involving the use of sets of orthogonal latin squares in "unorthodox" ways, and the other involving circular lattices in the plane. *D. R. Hughes.*

Ghosh, Birendranath. Optimum structure of rectangular sample units. Calcutta Statist. Assoc. Bull. **6** (1956), 176–180.

Rectangular sampling units (r.s.u.) are commonly used for studies on yield of a crop and on the proportion of land under a crop in both large-scale area sampling and in controlled experimental studies. The optimum structure of the r.s.u. has considerable practical interest. The structure of the r.s.u. is described in terms of the three elements (i) size $s=AB$, for $A\geq B=$width of r.s.u.; (ii) shape or elongation, $t=A/B$; and (iii) orientation, w, which is the direction in which the longer side, A, lies. The optimum structure of the r.s.u. is discussed briefly from bias and from efficiency considerations. Two situations, one involving no observational errors and the second involving observational errors, are considered. *W. T. Federer* (Ithaca, N.Y.).

Ghosh, Birendranath. A model for perimeter errors. Calcutta Statist. Assoc. Bull. **6** (1956), 189–192.

This paper represents an extension of the results of the paper reviewed above. The errors associated with a rectangular sampling unit (r.s.u.) are divided into interior and perimeter errors, which are assumed independent. A linear model is postulated for the errors along the perimeter of the r.s.u. The amount and nature of the bias due to perimeter errors is discussed; the effect of perimeter errors on the variance is examined. *W. T. Federer* (Ithaca, N.Y.).

Nordbotten, Svein. Allocation in stratified sampling by means of linear programming. Skand. Aktuarietidskr. **39** (1956), 1–6.

A number H of samples of size n_i $(i=1,\ \cdots,\ H)$ drawn from strata of size N_i are used to give the estimates $X_k'=\sum_{i=1}^{H}(N_i/n_i)\sum_{j=1}^{n_i}x_{kij}$ of the quantities

$$X_k=\sum_{i=1}^{H}\sum_{j=1}^{N_i}x_{kij}\quad(k=1,\ \cdots,\ K).$$

It is shown that if the cost of the experiment is

$$C_0+\textstyle\sum_{i=1}^{H}b_in_i^{-1}\quad(b_i\leq 0\text{ is realistic})$$

then the problem of choosing n_i so as to minimize the cost while keeping $\mathrm{var}(X_k')$ below arbitrarily prescribed upper bounds becomes, with the change of variable $y_i=n_i^{-1}$, a linear programming problem with K constraints. A numerical example $(H=K=2)$ is appended. *P. Wolfe.* (Princeton, N.J.).

Rudra, A. A method of discrimination in time series analysis. II. Sankhyā **17** (1956), 51–66.

To develop the Buys-Ballot table by variance component is a natural approach [see Rudra, Sankhyā **15** (1955), 9–34; MR **17**, 170], but the scanty numerical evidence available leaves it unclear to the reviewer how far the severe difficulties of a rigorous treatment are met

by the author's heuristic and approximate arguments.
H. Wold (Uppsala).

Horálek, Vratislav. Operating-characteristic curve for sampling inspection, where each product is checked for several independent quality characteristics. Apl. Mat. **1** (1956), 431–444. (Czech. Russian and English summaries)

The case of sampling inspection is considered, where each product is checked for k independent quality characteristics. A particular inspection scheme is set up for each quality characteristic. The resulting operating characteristic curve for such a procedure is developed under the assumption of mutual independence of individual quality characteristics. The operating characteristic curve must be determined by means of a band. Fundamental formulas for the bounds of the band, in which the operating characteristic curve is to lie, are given. The equations which determine the lower limit of the band are solved and a graphical method of solution proposed. The calculation of the limits is illustrated by a numerical example, in which the distribution of the number of defectives in a sample is approximated by Poisson's distribution.

The author plans to publish additional tables to Dodge-Romig's Tables of Sampling Inspection Plans in a further paper. These additional tables will be designed to enable the application of Dodge-Romig's tables to the case of acceptance sampling by a single sample, when several independent quality characteristics are checked.
J. Janko (Prague).

See also: Dykstra, p. 633; Linnik, p. 639; Benedetti, p. 651; Makover, p. 676; Ray, p. 678; Simaika, p. 679; Bellman, p. 681; De Groot and Mazur, p. 686; Burkhardt, p. 707; Haycocks and Plymen, p. 708; Krakau, p. 708; Robson, p. 708; Herdan, p. 708.

PHYSICAL APPLICATIONS

Mechanics of Particles and Systems

Vâlcovici, Victor. Les fondements de la Mécanique. Acad. R. P. Romîne. Stud. Cerc. Mat. **7** (1956), 373–398. (Romanian. Russian and French summaries)

Abzug, Malcolm J. Applications of matrix operators to the kinematics of airplane motion. J. Aero. Sci. **23** (1956), 679–684.

"The use of matrix transformations to represent finite rotations is fundamental to this paper. The concept was developed by Frazer, Duncan, and Collar [Elementary matrices $\cdots$, Cambridge, 1938]." The Euler (1776) matrix follows and is represented as a product of three matrices, each depending on only one Euler angle. The angular-velocity components on body-bound rectangular axes are expressed in terms of the Euler angles, the expressions being credited to Frazer, Duncan, and Collar [loc. cit.], Goldstein [Classical mechanics, Addison-Wesley, Cambridge Mass., 1950; MR **13**, 291], and Hall [Trans. Amer. Inst. Elec. Engrs. **69** (1950), 308–320]. Some other coordinate systems and angles are briefly considered. One page contains six plots of the inner and outer gimbal angles (appropriately defined) of vertical and directional gyros against the pitch, yaw, and bank (roll) angles of an airplane. *A. W. Wundheiler* (Chicago, Ill.).

de Simoni, Franco. Sulla geometrizzazione delle equazioni dinamiche di sistemi soggetti a vincoli anolonomi generali del prim'ordine. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. **90** (1956), 180–188.

The author studies a certain dynamical system S which is locally holonomic but is generally non-holonomic. If q^k are the Lagrangian coordinates of a point P in an n dimensional space Σ_n and if $v^k=\dot{q}^k$ represent the generalized velocity, then S is defined by a set of equations: $\phi_\alpha(q^k; v^k; t)=0$ such that $(\partial\phi_\alpha/\partial v^k)\delta q^k=0$, for $\alpha=1, 2, \cdots, m$. The kinetic energy T of S is given by the equation: $2T=a_{hk}v^hv^k+2a_hv^h+2a$, in which a_{hk}, a_h, and a are functions of the Lagrangian coordinates q^k and the time t. This system S is studied by use of the metric

$$ds^2=a_{hk}dq^hdq^k.$$

The Lagrangian equations of S are fully developed. It is found that the trajectories of the system S are the paths of autoparallelism of a space with the connections Γ_{hk}^i. These connections Γ_{hk}^i are of a complicated nature. Thus this space is in general, neither riemannian nor affine.
J. De Cicco (Chicago, Ill.).

Melis, Antonio. Un esempio di sistema anolonomo non lineare: il pattino guidato. Rend. Sem. Fac. Sci. Univ. Cagliari **25** (1955), 143–153 (1956).

This very interesting example of a nonholonomic dynamical system with nonlinear constraints may be described as two skate-like bodies hinged together and free to move on a smooth plane without side slipping. There are two constraint equations, one of which, applying only to one of the skates (the guiding skate), is the familiar linear one, while the other is quadratic in the velocity components of the guiding skate and the angular velocity of the other (so called guided) skate.
D. C. Lewis, Jr. (Baltimore, Md.).

Derwidué, L. Sur un théorème de Tait et Thomson. Mathesis **65**(1956), 196–201.

The theorem of Thomson and Tait is to the effect that a stable linear conservative system without gyroscopic forces remains stable after the introduction of dissipative and/or gyroscopic forces. This theorem is a translation of a purely algebraic theorem on the roots of a certain determinantal equation. Hence, contrary to the classical procedure, it is possible to prove it by purely algebraic methods. Such a proof (applicable also to a slightly more general problem) is presented by the author.
D. C. Lewis, Jr. (Baltimore, Md.).

Bottema, O. The Routh-Hurwitz condition for the biquadratic equation. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 403–406.

A short proof of the Routh-Hurwitz condition for stability in the special case of two degrees of freedom. The author considers geometrically the discontinuity arising in connection with the case of neutral stability, in which all four roots are pure imaginary. *D. C. Lewis, Jr.*

Garfinkel, Boris. On the motion of simple pendulum. Ordnance Computer Research Report, Ballistic Research Laboratories, Aberdeen Proving Ground, Md., vol. 3 (1956), no. 4, pp. 1–5. (Government Agencies, their contractors and others cooperating in Government research may obtain reports directly from the Ballistic

Research Laboratories. All others may purchase photographic copies from the Office of Technical Services, Department of Commerce, Washington 25, D.C.)

When, in a string-supported simple pendulum, the initial value of the energy lies in a certain range, the tension vanishes at some instant and the particle trajectory passes from a circular to a parabolic one. The paper discusses the nature and number of these transitions — an exercise in Lagrangian treatment of non-holonomic constraints. *W. Freiberger* (Providence, R.I.).

Rašković, Danilo P. On some characteristics of the frequency equations of small vibrations of some particular holonomic conservative systems. Quart. J. Mech. Appl. Math. **9** (1956), 334–344.

Analysis of the vibrations of some special linear systems characterized by the fact that the inertia matrix is diagonal and the stiffness matrix has non-zero elements only on the principal diagonal and the two diagonals adjacent thereto. *D. C. Lewis, Jr.* (Baltimore, Md.).

Fuller, A. T.; and Macmillan, R. H. Expressions for the damping and natural frequency of linear systems. Quart. J. Mech. Appl. Math. **9** (1956), 345–359.

By modifying the Routh-Hurwitz criteria three theorems about the roots of polynomial equations are derived. Theorem I locates the least negative of the real parts of the roots. Theorem II gives an approximate formula for this real part when its magnitude is small. Theorem III gives new expressions for the imaginary parts of the corresponding root-pair. Applications to the stability and frequency of linear systems are obvious. *D. C. Lewis, Jr.*

Basch, A. Eine konstruktive Bestimmung der Hauptrichtungen und Eigenfrequenzen der Schwingungen eines Systems von zwei Freiheitsgraden. Österreich. Ing.-Arch. **10** (1956), 119–124.

A geometrical method for the simultaneous diagonalization of two binary positive definite quadratic forms in the familiar dynamical context. *D. C. Lewis, Jr.*

Gröbner, W. Über die Berücksichtigung der Reibung bei Schwingungsproblemen. Österreich. Ing.-Arch. **10** (1956), 171–175.

The author uses variational methods for introducing a simplified practically satisfactory treatment of the complicated phenomenon of friction. *D. C. Lewis, Jr.*

Masotti, Arnaldo. Sul pendolo quadrantale. Rev. Un. Mat. Argentina **17** (1955), 117–124 (1956).

The quadrantal pendulum is a dynamical system governed by the equation $\varphi''=-c^2\sin\varphi\cos\varphi$, where φ is an angular coordinate of immediate significance. The substitution $\theta=2\varphi$ reduces the problem at once to that of the ordinary pendulum. The quadrantal pendulum occurs in special situations in rather advanced theories involving rigid bodies and hydrodynamics or electromagnetism. The author gives a series of examples in particle dynamics leading to the same equation.

D. C. Lewis, Jr. (Baltimore, Md.).

Johnson, D. C.; and Bishop, R. E. D. The modes of vibration of a certain system having a number of equal frequencies. J. Appl. Mech. **23** (1956), 379–384.

The conservative system considered has kinetic energy

$$T=\tfrac{1}{2}(M\dot{q}_0^2+m\dot{q}_1^2+\cdots+m\dot{q}_n^2)$$

and potential energy $V=\frac{1}{2}k\sum_{i=1}^{n}(q_0-q_i)^2$. The system thus has $(n+1)$ degrees of freedom and $(n+1)$ principal modes of vibration. Since the frequencies of these modes are not all distinct, the manner of choosing the modes is to some extent arbitrary. A method is given for selecting the modes so that certain conditions of orthogonality may be satisfied. *D. C. Lewis, Jr.* (Baltimore, Md.).

Fogagnolo Massaglia, Bruna. Sulle vibrazioni forzate di un sistema dissipativo, soggetto ad una forza di richiamo ritardata. Univ. e Politec. Torino. Rend. Sem. Mat. **15** (1955–56), 343–349.

A discussion of the equation,

$$\ddot{q}(t)+c\dot{q}(t)+\sigma^2q(t-a)=f(t)$$

with $a>0$, under the initial conditions $q(0)=\dot{q}(0)=0$. An explicit formula for a solution is given; but this formula is based on the assumption that $q(t)\equiv0$ for $-a\leq t<0$ and the "solution" given by the formula has therefore a discontinuous second derivative (if not actually undefined) at $x=0$, at least assuming $f(t)$ to be continuous and $f(0)\neq0$. Similar difficulties to be expected at points $x\equiv0 \pmod a$ are not discussed. *D. C. Lewis, Jr.* (Baltimore, Md.).

Leitmann, G. A calculus of variations solution of Goddard's problem. Astronaut. Acta **2** (1956), 55–62.

R. H. Goddard in a pioneering paper [Smithsonian Inst. Publ. Misc. Coll. **71**, no. 2 (1919)] considered the problem of sending a rocket to high altitude in the most economical way. Goddard idealized the problem retaining what he considered to be the important characteristics of the physical situation. The resulting formulation as an exercise in the Calculus of Variations, he found intractable, and so was content with an approximate solution by a numerical method of short arcs. This author finds an explicit closed-form solution (of rather intricate form) under simplifying assumptions, such as that of Newtonian drag law, and checks this formula by comparison with some special known results that follow when stringent particular conditions are imposed. Despite the satisfaction of thus triumphantly solving a mathematical problem, this reviewer prefers Goddard's own highly flexible, and refreshingly modern program permitting full utilization of all available data, and eminently suited to numerical computation by high-speed computers.

A. A. Bennett (Providence, R.I.).

Nelson, Robert L. The motions of rolling symmetrical missiles referred to a body-axis system. NACA Tech. Note no. 3737 (1956), 51 pp.

"In this paper the equations of motion referred to the body-axis are again derived, and assumptions similar to those of Nicolaides (who referred his equations to space axes) are used. Some of the possible missile motions are shown, together with the motions to be expected from certain forcing functions. Finally, the derived equations of motion are used to establish a technique for the reduction of oscillation data to obtain aerodynamic derivatives. The method is applied to the experimental results obtained from a rolling symmetrical missile."

R. A. Rankin (Glasgow).

See also: Diliberto and Hufford, p. 653; Miller, p. 658; Grabar', p. 662; Coleman and Feingold, p. 684; Loedel, p. 703.

Statistical Mechanics

Dutta, M. An essentially statistical approach to thermodynamic problem. II. Proc. Nat. Inst. Sci. India. Part A. **21**(1955), 373–381 (1956).

Part I of this paper [same Proc. **19** (1953), 109–126; MR **15**, 85] concerned a system composed of matter of a single chemical type. This Part deals with systems of the following types: free radiation, a mixture of radiation and matter of one chemical type, and matter of chemically different types. The usual formulae pertaining to variation of environment and laws of microscopic distributions are obtained. *C. C. Torrance* (Monterey, Calif.).

Kuznecov, P. I.; Stratonovič, R. L.; and Tihonov, V. I. Correlation functions in the theory of Brownian motion. Generalization of the Fokker-Planck equation. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **26** (1954) 189–207. (Russian)

De Groot, S. R.; and Mazur, P. On the statistical basis of Onsager's reciprocal relations. Physica **23** (1957), 73–81.

Author derives Onsager's reciprocal relations in two ways: (1) for a non-equilibrium distribution function of the form

$$f(\alpha, t)=f^e(\alpha)\{1+\varphi(\alpha, t)\},$$

where f^e is an equilibrium distribution function of Gaussian form, and φ is assumed (without justification) to be linear in the (even) state vector α. (2) A conventional derivation is outlined for equilibrium conditions. These two derivations are compared as to (a) assumptions made, and (b) naturalness and immediacy of implications, such as the Onsager relations and the H-theorem. A good case is made for the first derivation (except for the essential detail concerning φ). *C. C. Torrance.*

Counson, J.; Ledoux, P; et Simon, R. Viscosité et oscillations d'étoiles gazeuses. Bull. Soc. Roy. Sci. Liège **25** (1956), 144–162.

The coefficient of viscosity of ionized hydrogen is computed by the standard formulae of the Chapman-Enskog theory [Chapman and Cowling, The mathematical theory of non-uniform gases, 2nd ed., Cambridge, 1953, § 9.84; for a review of the 1st ed. see MR **1**, 187] on two different assumptions regarding the inter-atomic forces. The first assumption is the usual one of a Coulomb field which is cut off at a distance $\sim e^2/4kT$; and the second assumption is that of a Coulomb field screened by a uniform distribution of charges up to a certain distance, r_0, and an exponentially falling off field (like $e^{-\beta r}/r$) beyond r_0. The relevant "phase-shift" integrals are evaluated and it is shown that for typical stellar conditions the two assumptions give very concordant results for the coefficient of viscosity. The dissipation of energy which the deduced values of the coefficient of vicsosity imply is evaluated; its importance for the damping of stellar pulsations is shown to be negligible even though the new values of the viscosity are some 200 times larger than those which have been previously estimated. *S. Chandrasekhar.*

Raman, C. V. Quantum theory and crystal physics. Proc. Indian Acad. Sci. Sect. A. **54** (1956), 361–366.

The fundamental notions of quantum theory and thermodynamics indicate that a crystal should be regarded as an assembly of an immense number of oscillators whose energy states are quantised and which form a system in thermodynamic equilibrium. They also indicate that the spectroscopic properties and the thermal behaviour of crystals stand in the closest relation to each other. One is thus left with the problem of discovering and enumerating the oscillators of the different sorts comprised in the crystal and of determining their scheme of energy levels. This may be done by methods analogous to those which have proved successful in the field of molecular spectroscopy. The results obtained are in perfect agreement with the observed spectroscopic properties and thermal behaviour of crystals. *W. Nowacki* (Bern).

Wilson, A. J. C. The probability distribution of X-ray intensities. I. Acta Cryst. **2** (1949), 318–321.

The structure amplitude of the hkl reflection from a crystal containing N atoms per unit cell is given by

$$F=\sum_{j=1}^{N} f_j \exp\left[2\pi i(hu_j+kv_j+lw_j)\right],$$

where f_j is the scattering factor and u_j, v_j, w_j are the fractional coordinates of the jth atom. Then $I=|F|^2$ is called the intensity of this reflection. The author uses the central limit theorem to show that in a crystal without symmetry and containing a sufficiently large number of atoms the probability of a particular reflection having an intensity between I and $I+dI$ is $P(I)dI$, where $P(I)=\Sigma^{-1}\exp(-I/\Sigma)$ and $\Sigma=\sum_1^N f_j^2$. If the crystal has a center of symmetry the structure amplitude is real and, denoting by $P(F)dF$ the probability that the amplitude lie between F and $F+dF$ he finds

$$P(F)=(2\pi\Sigma)^{-\frac{1}{2}}\exp\left(-F^2/2\Sigma\right).$$

The author points out that the difference between these two probability distributions may serve as a basis for the purely x-ray determination of the existence of a center of symmetry in a crystal. In both cases the mean value of I is Σ, a fact that can be used to put experimentally observed relative intensities on an absolute basis. *H. A. Hauptman* (Washington, D.C.).

Wilson, A. J. C. The probability distribution of X-ray intensities. III. Effects of symmetry elements on zones and rows. Acta Cryst. **3** (1950), 258–261.

{Editor's note: The second paper of this series is of purely physical interest.}

The probability distribution of the intensity of a special hkl reflection, i.e. one for which one of h, k, l vanishes (zone reflection) or for which two of h, k, l vanish (row reflection), depends upon the symmetry of the crystal. The effect of the various symmetry elements (mirror plane, rotation axis, etc.) on these distributions is described. A symmetry element that causes equivalent atoms to coincide in groups of n in a plane (or line) projection produces a zone (or row) of reflections whose average intensity is $n\Sigma$. The difference between the zone (or row) average of intensities and the general average (i.e. Σ) may be useful in the purely x-ray determination of space groups. *H. A. Hauptman* (Washington, D.C.).

Rogers, D. The probability distribution of X-ray intensities. IV. New methods of determining crystal classes and space groups. Acta Cryst. **3** (1950), 455–464.

The differences among the distributions of certain sets of intensities are, as described in part III, characteristic of the symmetry elements present in the crystal. This

fact serves as the basis for identifying 215 of the 219 space groups by means of x-ray intensities alone.

H. A. Hauptman (Washington, D.C.).

Rogers, D.; and Wilson, A. J. C. The probability distribution of X-ray intensities. V. A note on some hypersymmetric distributions. Acta Cryst. **6** (1953), 439–449.

The repetition of a structural unit or motif involving the introduction of non-crystallographic symmetry, e.g. the existence of one or more noncrystallographic (i.e. local) centers of symmetry in a centrosymmetric space group (hypercentrosymmetry) or of two or more parallel non-centrosymmetric motifs (hyperparallelism), has a characteristic effect on the probability distribution of the magnitude (or intensity) of a structure amplitude. The authors consider several types of such pseudosymmetry and obtain the corresponding probability distributions. Usually different kinds of pseudosymmetry lead to different distributions, but not always. Hence, examination of the observed intensities of the structure amplitudes often enables one to detect the existence and type of pseudosymmetry. *H. A. Hauptman.*

Rogers, D.; Stanley, E.; and Wilson, A. J. C. The probability distribution of intensities. VI. The influence of intensity errors on the statistical tests. Acta Cryst. **8** (1955), 383–393.

Errors in the observed intensities of the structure amplitudes cause the observed distribution of these intensities to differ from the ideal (or theoretical) distribution and hence limit the usefulness of these distributions in detecting the presence of a center or plane of symmetry. The effects of four types of error are considered. The authors obtain results which enable one to estimate a likely outer limit for departures from the ideal statistical criteria correlated with a symmetry center or plane in any problem. Hence the basis is established for deciding whether the statistical tests for determining the presence of these two symmetry elements may be validly applied in any given case. *H. A. Hauptman.*

Wilson, A. J. C. The probability distribution of X-ray intensities. VII. Some sesquisymmetric distributions. Acta Cryst. **9** (1956), 143–144.

The introduction of one or more non-crystallographic centers of symmetry, in addition to the crystallographic center, results in an increase in the variance of the probability distribution of the intensity of a structure amplitude [cf. part V]. The addition of new crystallographic symmetry elements, such as one or more mirror planes, causes a reduction in the variance to a value intermediate between those of the centrosymmetric and appropriate hypercentrosymmetric cases.

H. A. Hauptman (Washington, D.C.).

Chandrasekhar, S. A first-order correction for extinction in crystals. Acta Cryst. **9** (1956), 954–956.

It is shown that the various formulae for the integrated intensity of an X-ray reflexion affected by extinction [cf., e.g., H. Ekstein, Phys. Rev. (2) **83** (1951), 721–729; MR **13**, 414; A. R. Lang, Proc. Phys. Soc. Sect. B. **66** (1953), 1003–1008] can be expressed to a first approximation in the form

$$\rho_\perp = \alpha|F|^2 - \beta|F|^4,$$

$$\rho_\parallel = \alpha|F|^2\cos^2 2\theta - \beta|F|^4\cos^4 2\theta,$$

where $\rho_\perp$ and $\rho_\parallel$ are respectively the intensity of reflexion of X-rays polarized normal and parallel to the plane of incidence, θ is the angle of reflexion, and F is the structure amplitude. The quantity α can be calculated for any given experimental arrangement, β depends on the type of extinction (primary, secondary, or both), but not on F or θ. By elimination of β between the equations above, relative values of $|F|^2$, corrected for extinction, can be obtained as

$$|F|^2 = \frac{\rho_\parallel - \rho_\perp\cos^4 2\theta}{\alpha(\cos^2 2\theta - \cos^4 2\theta)},$$

where $\rho_\parallel$, $\rho_\perp$ and θ are experimentally measurable quantities. *A. J. C. Wilson* (Cardiff).

Tenerz, E. Remarks on the calculation of characteristic temperatures of cubic crystals from the elastic constants. Ark. Fys. **11** (1956), 247–252.

The Debye characteristic temperature Θ_D for cubic crystals is calculated from the elastic constant, using the method first introduced by Bhatia and Tauber [Phil. Mag. (7) **45** (1954), 1211–1213]. These authors expand the mean velocity of the three waves in Kubic Harmonics K_m of which they take only three into account. Extending that method by taking five K_m, of which the fifth is deduced here, a relative simple expression for Θ_D is still obtained. It is accurate even for large elastic anisotrophy. A numerical comparison with Quimby and Sutton's method [Phys. Rev. (2) **91** (1953), 1122–1127] shows a very good agreement in the cases where this method is convenient to use. The numerical work is almost the same for both methods in these cases, otherwise much less when Kubic Harmonics are used. Values of Θ_D for a number of substances and temperatures are calculated. The new calculation of Θ_D for KCl at 0°K exhibits good agreement with measurements in the T^3-range. The fifth K_m can be of interest in other calculations.

W. Nowacki (Bern).

Harrison, Walter A. Scattering of electrons by lattice vibrations in nonpolar crystals. Phys. Rev. (2) **104** (1956), 1281–1290.

A theoretical analysis of the scattering of electrons by lattice vibrations in nonpolar crystals is made which permits an estimate of the relative importance of acoustical-mode and optical-mode scattering in monatomic nonmetals. This comparison is achieved by expanding the matrix elements for scattering in powers of the electron wave number. Individual terms are examined in the light of crystal symmetries and the nature of the band minimum associated with the carriers. The order of the first nonvanishing term is obtained in each case. The temperature dependence of the mobility as indicated by these results is compared with that observed in the cases of electrons and holes in silicon and germanium. The observed dependence upon temperature is understandable in terms of the expected relative importance of acoustical- and optical-mode scattering in the cases of holes in silicion and germanium and electrons in germanium. In the case of electrons in silicon, optical modes are not expected to contribute appreciably and the strong temperature dependence must arise from some other mechanism, presumably intervalley scattering. The relation between the results obtained here and those of deformation-potential theory is discussed briefly. *W. Nowacki* (Bern).

See also: Murty, p. 701; Markham, p. 702.

Elasticity, Visco-elasticity, Plasticity

Rozovskiĭ, M. I. Semisymbolic method of solving certain problems of the theory of creep. Akad. Nauk Armyan. SSR. Izv. Fiz. Mat. Estest. Tehn. Nauki **9** (1956), no. 5, 43–60. (Russian. Armenian summary)

Complex creep cannot be described in terms of a finite number of time dependent parameters. Symbolic solutions of complex creep problems for linear stress-strain relation were given by V. Volterra [Leçons sur les fonctions de lignes, Gauthier-Villars, Paris, 1913] and Yu. N. Rabotnov [Prikl. Mat. Meh. **12** (1948), 53–62; MR **9**, 546]. Symbolic solutions are obtained from Volterra integral representation of after-effects where in Hooke's law elastic constants are replaced by appropriate integral operators. Symbolic method introduces initially a special function of time and space variables. Semisymbolic method used by the author of this paper does not require initial introduction of new functions. It can be applied to linear theory (linear stress-strain relation) and also to certain special cases of non-linear theory. The author gives the following solutions of creep problems obtained from the solution of an integro-differential equation: 1) a linear case illustrated on a cylindrical tank under hydrostatic pressure, 2) a non-linear case illustrated on a beam, 3) a dynamic case illustrated on vibrations of an unbounded medium where creep is taken into account.

T. Leser (Aberdeen, Md.).

Wolibner, W. Sur le mouvement des corps friables. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 507–509.

Author studies motion of friable (i.e., easily crumbled or pulverized) bodies based on admittedly insufficiently substantiated assumptions using classical tensorial fluid mechanics concepts. He considers both static and sliding friction and heuristically investigates special cases.

W. J. Nemerever (Seattle, Wash.).

Froyd, R. K. Note on a problem considered by Tiffen. Quart. Appl. Math. **14** (1957), 426–428.

This is a short paper to show that the results of Tiffen for the stress distribution in a semi-infinite plane with a parabolic boundary [Quart. J. Mech. Appl. Math. **5** (1952), 352–360; MR **14**, 333] may be deduced at once from the results of Muschelisvili [Some basic problems of the mathematical theory of elasticity, 3rd ed., Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1949, § 95; MR **11**, 626; **15**, 370].

R. M. Morris (Cardiff).

Stadelmaier, Hans H. Spannungsfeld einer auf den Rand einer Halbebene wirkenden Einzellast bei elastischer Anisotropie. Z. Angew. Math. Phys. **7** (1956), 393–402.

The solution of the two-dimensional stress problem in an anisotropic medium is given in closed form for the boundary conditions of a concentrated load at a point on the straight boundary of a semi-infinite plate. The properties of a periodic repetition of such concentrated loads are discussed and compared with an existing observation of a non-uniform stress distribution in α-iron. (Author's summary.)

W. Nowacki (Bern).

Koževnikova, V. N. Stress distribution around a rectangular opening in an infinite plate deformable in its own plane. L'vov. Gos. Univ. Uč. Zap. **29**, Ser. Meh.-Mat. no. 6 (1954), 112–130. (Russian)

The author investigates the stress distribution in an infinite plate and an infinite strip bent by moments acting in its plane, in the presence of a small opening whose shape closely approaches that of a rectangle. It is assumed that the opening is located symmetrically with respect to the neutral axis.

Mushelisvili's method is used in the solution of the problem. The given plate (the z-region) is conformally mapped into the unit circle (the ζ-region) and the complex stress functions $\varphi(\zeta)$ and $\psi(\zeta)$ are obtained for this region. The stresses found with the help of these functions are then translated back into the z-plane. The mapping function and the stress functions $\varphi(\zeta)$ and $\psi(\zeta)$ are assumed in the form of Laurent expansions with a finite number of terms. These stress functions satisfy exactly the boundary conditions at the edge of the opening. If the plate has a finite width, the boundary conditions at the outer edges of the plate are not fully satisfied. However, the author shows that for a plate whose width is approximately three times the width of the opening, the error thus introduced is negligibly small.

The stress distribution at the edge of the opening and the maximum values of the stress concentration factors for openings of different side ratios are given. The highest stress occurs at the corners of the opening, and the value of the stress concentration factor varies between 2.2 and 3.8.

The author mentions that photoeleastic stress measurements performed by others give results which agree well with the theory.

H. V. Hahne (Palo Alto, Calif.).

Prager, William. A new method of analyzing stresses and strains in work-hardening plastic solids. J. Appl. Mech. **23** (1956), 493–496.

A piecewise linear loading function for a strain-hardening material based on the hexagonal Tresca-St. Venant yield-condition that is displaced in proportion to the total plastic strain within the σ_1—σ_2-plane is proposed in the form of an ingenious mechanical model, and used for the solution of the problem of the circular plate of rigid-strain-hardening material.

A. M. Freudenthal.

Gasser, Hans-H. Knicken eines gestreckten Gelenkstabzuges. Z. Angew. Math. Phys. **8** (1957), 64–71.

Die Arbeit untersucht folgendes Problem: Wird ein Stab zentrisch gedrückt, knickt er unter einer bestimmten kritischen Last aus. Dabei kann die Biegelinie einen Wendepunkt haben, was bedeutet, dass an dieser Stelle während des Knickens kein Biegemoment entsteht. Es kann daher gefragt werden, ob an dieser Stelle ein Gelenk angeordnet werden darf, ohne dass dadurch die Stabilität geändert wird. Die Untersuchung zeigt, dass dies in einem Fall tatsächlich möglich ist, in anderen Fällen dagegen nicht.

R. Gran Olsson (Trondheim).

Guest, J. A lower-bound solution to the buckling stress of uniformly clamped parallelogram plates. Austral. J. Appl. Sci. **7** (1956), 336–345.

The theory for the lower limit of the buckling load of flat rectangular plates given by E. Trefftz [Z. Angew. Math. Mech. **15** (1935), 339–344] has been improved by B. Budiansky, Hu, and Connor [NACA Tech. Note nos. **1103** (1946); **1558** (1948); MR **8**, 118; **9**, 481]. Following these papers, the author extends the study of the buckling of clamped rectangular plates in compression to the case of parallelogram plates. The Lagrangian multiplier method can be used to find both upper and lower bounds to the critical buckling stress of a uniformly compressed

clamped parallelogram plate, but the computations involved for the upper bound compare rather unfavourably with the usual Rayleigh-Ritz method. For that reason these results are not discussed in the paper. (In the second inequality on p. 336 there is a misprint.)

R. Gran Olsson (Trondheim).

Klein, Bertram. The buckling of tapered plates in compression. Compressive buckling under varying loading of simply supported plates simultaneously tapered in plan form and in thickness. Aircraft Engrg. **28** (1956), 427–430.

A simple matric formulation is developed for calculating the elastic buckling of a class of simply supported plates that are simultaneously tapered in plan form and in thickness in one direction and subjected to bi-axial loading that varies in each direction. Design curves are presented for the special case when the thickness taper is linear and the plate material is equally stressed in the direction of this thickness taper. *R. Gran Olsson.*

Conroy, M. F. The elastic stresses at the boundary of a symmetrically shaped hole in an infinite plate loaded by normal boundary forces in the plane of the plate. Bull. Calcutta Math. Soc. **48** (1956), 47–54.

In the present paper the elastic stresses at the boundary of a symmetrically shaped hole in an infinite plate, due to bands of force or concentrated forces applied normal to the boundary and in the plane of the plate, are determined. The class of hole shapes considered are those contours in the z-plane into which the unit circle in the ζ-plane is mapped by the function

$$z=\zeta(A_0+\sum_{n=1}^{n=m} A_n\zeta^{-kn}),$$

where $k\geqq 2$ and the quantities A_n are real coefficients restricted so that the roots of $dz/d\zeta$ lie within the unit circle in the ζ-plane. The force distribution has an angular periodicity of $2\pi/k$. The method of solution consists of a combination of the conformal mapping technique and the method of N. I. Muskelisvili [Some basic problems of the mathematical theory of elasticity, 3rd ed., Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1949, ch. 15; MR **11**, 626; **15**, 370; see also I. S. Sokolnikoff, Mathematical theory of elasticity, 2nd ed., McGraw-Hill, New York, 1956, pp. 249–327; MR **17**, 800] for solving plane elasticity problems. The results obtained are applicable to finite plates to such an extent that the stresses at the outer boundary due to the loading at the inner boundary are negligible.

The boundary stresses were derived for a wide and practical class of boundary contours. It was found that these stresses assume a very simple form in the case of a hole in the shape of a hypotrochoid loaded by periodically spaced concentrated forces. *R. Gran Olsson.*

Sharma, Brahmadev. Thermal stresses in infinite elastic disks. J. Appl. Mech. **23** (1956), 527–531.

The author presents the general formulation of the thermal stresses problem as given by S. Timoshenko and J. N. Goodier [Theory of elasticity, McGraw-Hill, New York, 1951; MR **13**, 599] and solves at a steady state the following cases: 1) an isotropic infinite plate with axi-symmetric temperature distribution, 2) a semi-infinite solid in which the infinite plate from 1) is the upper part and on the lower boundary of the plate takes place a discontinuous change of elastic constants, 3) a semi-infinite solid where a circular portion of the boundary is kept at a constant non-zero temperature and the remaining part of the boundary is kept at zero temperature. In the second section of the paper the author presents a general solution at an unsteady state.

T. Leser (Aberdeen, Md.).

Chakravorty, J. G. Some problems of torsion of spherically aeolotropic bodies. J. Assoc. Appl. Phys. **3** (1956), 61–68.

The present paper contains a discussion of the general case of torsion of a solid of revolution and solutions have been obtained in some particular cases.

The author previously [same J. **3** (1956), 14–16; MR **18**, 163] solved the problem of twisting of a spherical shell of spherically aeolotropic material by tangential forces and obtained expressions for tractions on the upper and lower hemispherical surfaces. In the present paper the general case of torsion of a solid of revolution of spherically aeolotropic material has been discussed and particular solutions have been obtained in the case of (a) torsion of a cone, (b) torsion of an incomplete spherical shell, and (c) complete spherical shell twisted by surface forces, all these bodies being of spherically aeolotropic material. Similar problems for isotropic solids have been solved by A. E. H. Love Mathematical Theory of Elasticity, New York 1944, p. 298–300. *R. Gran Olsson.*

Longuet-Higgins, H. C. The vibrations of a stressed framework. Phil. Mag. (7) **46** (1955), 98–100.

An elegant vectorial derivation of the equations for the normal modes of vibration of the most general system of mass particles connected in pairs by light springs.

D. C. Lewis, Jr. (Baltimore, Md.).

Mindlin, R. D. Simple modes of vibration of crystals. J. Appl. Phys. **27** (1956), 1462–1466.

Here are described the solutions of the three-dimensional equations of elasticity and the equations of generalized plane stress pertaining to vibrations of crystal rectangular parallelepipeds and thin plates with traction-free faces. The modes of motion exist in crystals of certain classes for specific orientations and length-ratios of the edges. Since the solutions are exact and in closed form, they may be used to advantage in connection with the measurement of elastic constants. Special disposition of electrodes is required for piezoelectric excitation. *W. Nowacki* (Bern).

Coleman, Robert P.; and Feingold, Arnold M. Theory of self-excited mechanical oscillations of helicopter rotors with hinged blades. NACA Tech. Note no. 3844 (1957), vii+131 pp.

During the 2nd World War, some helicopters were observed during ground tests to exhibit a violent oscillatory rotor instability, which was at first attributed to rotor-blade flutter, but a careful analysis indicated it to be caused by an unknown phenomenon in which the rotational energy of the rotor was converted into oscillatory energy of the blades. From 1942 to 1947 NACA made an effort to explain this phenomenon. These results are realised in this book, which consists of three separate papers with four appendices and 31 figures. In the first paper "Theory of self-excited mechanical oscillations of hinged rotor blades" (by R. Coleman), one gives the approach to the vibration problem, mathematical developments and method of applying theory in the case

when the damping is or is not neglected. A very interesting idea in the rotor vibration theory is the existence of the polar symmetry. When this symmetry exists then the solution can be expressed in terms of exponential or trigonometric functions, in the opposite case one obtains the Mathieu's functions. Important practical examples are discussed.

The second chapter "Theory of mechanical oscillations of rotors with two hinged blades" (by A. Feingold) deals with the mathematical analysis and discussion of the frequency equations and vibrating modes which are generally elliptical. It is shown that there exist six resonant frequencies and that the instability region is bounded by two shaft critical speeds.

The next chapter "Theory of ground vibrations of a two-blade helicopter rotor on anisotropic flexible supports" (by Coleman-Feingold) is an extension of previous work in the case of one or two blades on unequal supports. Mathematically, the problem reduces to differential equations with periodic coefficients. The existence of an infinite number of instability ranges is shown and results are discussed.

The exposition is concise and clear with very well drawn figures and with experimental results of tests of a simple model that is very useful in technical practice.

D. Rašković (Belgrade).

Hasselgruber, H. Zur Berechnung der Eigenfrequenzen eines in seiner Ebene frei schwingenden, nicht geschlossenen Kreisringes konstanten Querschnittes. Forsch. Gebiete Ingenieurwesens **22** (1956), 158–166.

The first few characteristic frequencies of a thin open circular ring, free to move in a plane, are computed by the Rayleigh-Ritz method. The main point of interest is the occurrence of integral conditions in addition to the normal boundary conditions. These integral conditions express the vanishing of the velocity of the center of gravity as well as the vanishing of the total angular momentum. Without these integral conditions the problem would be semidefinite, that is it would admit the eigenvalue zero. *D. C. Lewis, Jr.* (Baltimore, Md.).

Volterra, E. The equations of motion for curved and twisted elastic bars deduced by the use of the "Method of internal constraints". Ing.-Arch. **24** (1956), 392–400.

The purpose of the present paper is to show how the method of internal constraints, which has been used in a previous paper [Ing.-Arch. **23** (1955), 402–409; MR **17**, 556] for deriving the equations of motion of curved elastic bars, can also be applied for deriving the equations of motion for curved and twisted bars taking into account the effects of shear stress and of rotatory inertia.

The problem of vibrations of curved and twisted bars has many important applications among which are the vibrations of helicoidal springs and the vibrations of turbine buckets. *R. Gran Olsson* (Trondheim).

Hill, R. On the problem of uniqueness in the theory of a rigid-plastic solid. II. J. Mech. Phys. Solids **5** (1956), 1–8.

In a previous paper [same J. **4** (1956), 247–255; MR **18**, 83], the author has discussed the general problem stated in the title. In this part, specific application to the torsion of a cylinder is considered. The velocity components are assumed to be of the St. Venant form. The stress equations are referred to curvilinear coordinates (the direction of the shear-stress vector and the characteristic direction). The locus Γ of the intersections of the characteristics plays an important role in the subsequent analysis. The condition that the plastic work rate be positive determines the sense of the positive arc lengths along all characteristics which intersect Γ. The state of stress can then be determined from the resulting equations.

The warping rate can be found by first noting that the strain rate must be pure shear over the cross section and must have the same direction as the shear stress. The characteristics coincide with those of the stress-field. If all the characteristics cut Γ, the yield point distribution of stress and the velocity field are uniquely determined. Otherwise, there is a single infinity of virtual modes. Using the uniqueness criteria of Part I, the author then shows how the actual strain-rate and stress-rate fields may be selected from the virtual modes previously found. The process is illustrated in the case of a hollow section with uniform wall thickness. *G. H. Handelman.*

Thomas, T. Y. On the propagation of weak discontinuities in perfectly plastic solids. J. Math. Mech. **6** (1957), 67–85.

For the von Mises and Prandtl-Reuss equations of plasticity, the author gives a detailed analysis of surfaces singular of order one relative to the density, velocity and stress. Possible speeds and directions of such surfaces are determined. One of the more interesting results concerns the von Mises equations. For these, there exists the possibility of waves not parallel to directions of maximum or minimum shearing stress, a situation which is impossible for other types of waves which have been analyzed.

J. L. Ericksen (Washington, D.C.).

Parsons, D. H. Plastic flow with axial symmetry, using the Mises flow criterion. Proc. London Math. Soc. (3) **6** (1956), 610–625.

Author shows how by a suitable choice of new variables the axisymmetric problem for a perfectly plastic von Mises material can be reduced from the standard formulation in terms of seven dependent variables to a system of four quasi-linear first order partial differential equations which is immediately seen to possess no real characteristics — as was indeed shown by P. S. Symonds [Quart. Appl. Math. **6** (1949), 448–452; MR **10**, 759]. Some of the more usual boundary conditions are formulated in terms of the new variables but no attempt is made to solve a boundary value problem.

Attention may be drawn to a paper by R. T. Shield [Proc. Roy. Soc. London. Ser. A. **233** (1955), 267–287; MR **17**, 687] in which the same problem is discussed for a material obeying the Tresca yield condition and associated flow rule and solutions are obtained for a variety of problems. *W. Freiberger* (Providence, R.I.).

Freiberger, W. Bounds for the collapse load of a beam compressed by three dies. Austral. J. Phys. **9** (1956), 419–424.

The present paper deals with the problem of the plastic deformation of a beam under the action of three perfectly rough rigid dies, two dies applied to one side, one die to the other side of the beam, the single die being situated between the two others. The problem is treated as that of plane plastic flow. Discontinuous stress and velocity fields are assumed and upper and lower bounds for the pressure sufficient to cause pronounced plastic

yielding determined by limit analysis. As pointed out by the author the treatment is approximate and subject to experimental verification. *R. Gran Olsson.*

See also: Scotto Lavina, p. 676; Hersch, p. 678; Mirzoyan, p. 692; Kasahara, p. 705; Cypkin, p. 709.

Fluid Mechanics, Acoustics

Milne-Thomson, L. M. A general solution of the equations of hydrodynamics. J. Fluid Mech. **2** (1957), 88.

To write a review having at least the brevity of this elegant use of mathematics was not easy. The paper treats compressible or incompressible fluids of constant viscosity and finds the density, velocity and stress distribution in terms of an arbitrary function and a second rank arbitrary tensor. Even the uninitiated have but two simple referals to lean on [Milne-Thomson, Theoretical hydrodynamics, 3rd ed., Macmillan, New York, 1956; J. London Math. Soc. **17** (1942), 115–128; MR **4**, 122]. *M. G. Scherberg* (Dayton, Ohio).

Bryant, R. A. A. The one-dimensional and two-dimensional gas dynamics analogies. Austral. J. Appl. Sci. **7** (1956), 296–313.

The author investigates the possibility of using the flow of water in a channel, which has not got a rectangular cross-section, as an analogy to one- and two-dimensional gas flows. It is shown that in the one-dimensional case a channel of non-rectangular cross-section, of stated shape, does correctly give a flow which is mathematically analogous to a flow of gas, whose ratio of specific heats (γ) is 1.4. (In the one-dimensional case a rectangular cross-section gives a fictitious $\gamma=2$.) However, in the case of two-dimensional flows it is not possible to obtain a mathematical analogy unless the channel has a rectangular section, and in this case γ has the fictitious value of 2. *G. N. Lance* (Southampton).

Long, Robert R. Sources and sinks at the axis of a rotating liquid. Quart. J. Mech. Appl. Math. **9** (1956), 385–393.

L'Auteur forme une solution des équations du mouvement d'un liquide parfait, incompressible, en rotation dans un cylindre indéfini, en présence d'un puits sur l'axe de rotation.

L'allure de la solution dépend de la valeur du nombre de Rossby R_0 qui mesure le rapport des forces d'inertie et de Coriolis.

Il existe une valeur critique de R_0 au-dessous de laquelle il n'y a plus de solution du type envisagé.

Différentes photographies provenant d'une étude expérimentale mettent en évidence certains résultats de la théorie. *R. Gerber* (Toulon).

Bagnold, R. A. The flow of cohesionless grains in fluids. Philos. Trans. Roy. Soc. London. Ser. A. **249** (1956), 235–297.

Le travail actuel est consacré à l'écoulement non-uniforme des graines par-dessus une fondation sujet a l'action de la pésanteur. [Pour les contributions antérieures de l'auteur voir, e.g., Proc. Royal Soc. London. Ser. A. **157** (1936), 594–620; **225** (1954), 49–63.] Les propriétées dynamiques de ces types d'écoulement sont étudiées minitieusement. La méthode proposée par l'auteur est emperico-théorique. L'auteur n'établit pas aucunes équations générales de l'écoulement. *K. Bhagwandin.*

Žautykov, O. A. On a problem of filtration. Izv. Akad. Nauk Kazah. SSR. Ser. Mat. Meh. **1956**, no. 4(8), 70–79. (Russian)

The author considers filtration in a circular ring-shaped region. He obtains an explicit solution to the equation

$$\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}=\frac{1}{a^2}\frac{\partial p}{\partial r},$$

with the boundary-conditions

$$p(r,0)=\varphi(r),\quad p(R_k,t)=p_0=\text{const},$$

$$\alpha\frac{\partial}{\partial r}(p(R_l,t))=\beta\frac{\partial}{\partial t}(p(R_l,t)),$$

α, β, R_l, R_k are constants, $R_l\leq r\leq R_k$. A solution which satisfies all the boundary-conditions is found by means of the Laplace transform. The expressions seem a little complicated; nevertheless, numerical results can be obtained without great difficulties by means of modern digital computers. *K. Bhagwandin* (Oslo).

Ballabh, R. Steady superposable flows with cylindrical symmetry. Ganita **6** (1955), 15–21 (1956).

The author continues his studies of superposable fluid motions [J. Indian Math. Soc. (N.S.) **16** (1952), 191–197; MR **14**, 1029], this time considering the case of axially-symmetric flows of an incompressible non-viscous fluid. *J. B. Serrin* (Minneapolis, Minn.).

Finn, Robert. Some theorems on discontinuous plane fluid motions. J. Analyse Math. **4** (1955/56), 246–291.

This paper contains the proofs of results stated in an earlier note [Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 983–985, 985–987; MR **16**, 296]. The demonstrations are based on the well-known Levi-Civita representation for free boundary flows, and make use of the method of continuity. In addition to the work reported in the earlier note, the author also studies by the same method the classical problem of free-boundary flow past a polygonal obstacle. For this problem, a local uniqueness theorem is proved for concave obstacles (especially significant in that no restrictions are placed on total curvature), and an existence theorem is obtained for both symmetric and asymmetric convex obstacles, again with no restriction on curvature. Finally, the existence theorems are extended to apply to curved nozzles and obstacles. *J. B. Serrin* (Minneapolis, Minn.).

Broer, L. J. F. On the hydrodynamics of visco-elastic fluids. Appl. Sci. Res. A. **6** (1956), 226–236.

This paper refers to the equations of motion of the particular type of visco-elastic medium which is called a Maxwell body in the terminology of Reiner. In usual applications motion is very slow and inertia terms can be disregarded. When this condition does not apply, the momentum must be taken into account and the hydrodynamic equations must be formulated. These equations and the energy relations are established by a procedure analogous to the customary one except in that the Navier-Stokes equation is substituted by the more general characteristic equation of the Maxwell body. The equations are too complex to allow one to obtain exact solutions in the general case. So two characteristic partic-

ular cases are discussed. a) Flow periodic in time (steady state). The equations reduce to that of ordinary viscous flow by introduction of the complex coefficient of viscosity in the Navier-Stokes equation. As an example, laminar flow in a cylindrical tube is treated. b) Slow steady motion can be treated by an iteration process in which the first approximation is given by a solution neglecting inertia and elasticity (i.e. the solution of the corresponding problem in Stokes approximation). As an example steady two-dimensional flow is discussed. The importance of a characteristic number determining ratio of elastic to inertia forces is pointed out. *B. Gross.*

Srivastava, A. C. Beltrami motions in non-Newtonian fluids. J. Assoc. Appl. Phys. **3** (1956), 69–72.

In the terminology of this paper a non-Newtonian fluid is a fluid for which the stress tensor $\mathbf{T}$ and deformation tensor $\mathbf{D}$ are related by equation $\mathbf{T}=-p\mathbf{I}+2\mu\mathbf{D}+\gamma\mathbf{D}^2$, where μ and γ are coefficients of viscosity. The following results are proved concerning Beltrami motions ($\mathbf{v}\times \operatorname{curl}\mathbf{v}=0$) of an incompressible non-Newtonian fluid: 1) Beltrami motions are superposable and self-superposable in non-Newtonian liquids; 2) Any two motions in which the vortex lines of the first coincide with the stream lines of the second and the vortex lines of the second coincide with the stream lines of the first are superposable on each other; 3) The conditions of integrability of the equations of motion in the case of Beltrami motions in the non-Newtonian liquids are the same as those in Newtonian liquids. *J. B. Serrin.*

Lance, G. N.; and Deland, E. C. The shape of the nappe of a thin waterfall. Quart. J. Mech. Appl. Math. **9** (1956), 394–399.

L'Auteur forme l'équation différentielle du deuxième ordre dont la résolution donne le profil d'un jet plan, infiniment mince, d'un liquide pesant, lorsque l'on tient compte de la tension superficielle et de l'effet d'une différence de pression sur les deux faces du jet.

Cette équation se réduit à une équation du premier ordre, dont la solution ne peut s'expliciter au moyen de fonctions élémentaires, mais qui a été résolue sur l'analyseur différentiel de l'Université de Los Angeles. Les courbes qui définissent ce profil du jet sont données pour différentes valeurs des paramètres, mettant en évidence le rôle joué par la pression sur l'allure du profil.

R. Gerber (Toulon).

Stewart, Homer J.; and Ormsbee, Allen I. Conical techniques for incompressible nonviscous flow. J. Aero. Sci. **23** (1956), 1029–1036.

It is shown how certain useful solutions of the equations of inviscid incompressible flow in three dimensions can be constructed by superposition of homogeneous solutions of order zero. These homogeneous solutions are of the form

$$F(\varepsilon)+G(\bar{\varepsilon}),$$

where $\varepsilon=(y+iz)/(x+r)$ and $r=(x^2+y^2+z^2)^{\frac{1}{2}}$. Two such homogeneous solutions are used here; one represents a uniform source distribution over a conical portion of the xy-plane, the other an analogous uniform vortex distribution. The former was used by Jones [NACA Tech. Note no. **1340** (1947); MR **8**, 542] and the latter by Lampert [J. Aero. Sci. **18** (1951), 107–114]. They can be used to represent the flow about polygonal non-lifting wings and constant-pressure lifting wings, respectively, by means of superposition techniques. The paper includes some remarks indicating that certain of Lampert's solutions (loc. cit.) are in error.

W. R. Sears (Ithaca, N.Y.).

Leslie, D. C. M. An introduction to second-order wing theory. J. Aero. Sci. **24** (1957), 99–106.

The second order effect of a particular thickness distribution on the lift of a rectangular wing of effective aspect ratio greater than one (for which the side edges are independent) is calculated with the aid of the particular integral of Fell and Leslie [J. Aero. Sci. **21** (1954), 851–852; Quart. J. Mech. Appl. Math. **8** (1955), 257–265; MR **16**, 303; **17**, 314] and found to be unimportant (in two dimensions it vanishes identically). The more important problem of the pitching moment could be attacked along the same lines, but the algebra would be extremely involved.

The reviewer notes that the top curve in Fig. 3 should be labelled $N=2^{\frac{1}{2}}$ (not 2). *J. W. Miles.*

Sidorov, O. P. Flow about the simplest annular wing. Kazan. Aviac. Inst. Trudy **29** (1955), 9–24. (Russian)

The incompressible flow of a uniform stream past a ring is considered; the ring has a transverse plane of symmetry but otherwise its cross section can be of general shape. The uniform stream is at zero incidence so that no resultant circulation arises and vortex sheets are absent. The flow field, in particular, the surface velocity and pressure distribution are determined from a stream function expressed in terms of complete elliptic integrals of the third kind. An example is worked out. *M. Holt.*

Mirzoyan, A. A. Solution of two special problems of hydrodynamics of visco-plastic fluid. Izv. Akad. Nauk Azerbaidžan. SSR **1955**, no. 11, 13–18 (1956). (Russian. Azerbaijani summary)

The first problem is that of the steady motion of a visco-plastic fluid contained between two coaxial circular cylinders under the assumption that the inner cylinder moves with velocity v along its axis.

The second problem is that of the motion of a visco-plastic fluid bounded by a pipe of rectangular cross-section and a number of identical cylindrical heaters of circular cross-section. The heaters are arranged in equidistant columns and rows inside the rectangular pipe and parallel to its sides.

In both cases the differential equations of motion, with the corresponding boundary conditions, are solved and the rate of discharge of the fluid over the cross-section of the pipe is calculated. *E. Leimanis* (Vancouver, B.C.).

Yuan, S. W. Further investigation of laminar flow in channels with porous walls. J. Appl. Phys. **27** (1956), 267–269.

The author gives an asymptotic solution of the problem described in the title for moderately large velocities of infection or suction through the porous walls. The results agree with those obtained by other methods. The mathematical nature of the asymptotic solution is not discussed, even though there is an apparent necessity for doing so, as indicated by a reduction of order of the equation solved. *C. C. Lin* (Cambridge, Mass.).

Rusanov, B. V. Slow unsteady motion of a viscous liquid in a half-plane. Vestnik Leningrad. Univ. **11** (1956), no. 13, 109–118. (Russian)

The linearized Navier-Stokes equations for incom-

pressible plane flow become in proper units (*) $\partial w/\partial t = \Delta\bar{w} - \operatorname{grad} p$ and (**) $\partial u/\partial x + \partial v/\partial y = 0$, where $\bar{w}=(u, v)$. The author seeks a solution in $y \geqq 0$, $t \geqq 0$ such that $\bar{w}(x, y, 0)=0$ and $\bar{w}(x, 0, t)=\bar{w}^0(x, t)$. By assuming first that $\bar{w}^0(x+2l, t)=\bar{w}^0(x, t)$, constructing Fourier series solutions, and then passing to the limit as $l=\infty$, he obtains Fourier integral representations for $\bar{w}, p$. He proves that these $\bar{w}, p$ actually are the desired solution if (1) $\bar{w}^0$ is continuous except for a finite set of values $x=x_i$ independent of t, and $\partial\bar{w}^0/\partial x$ and $\partial\bar{w}^0/\partial t$ are continuous wherever $\bar{w}^0$ is; (2) there exists $N(x) \geqq \max(|\bar{w}^0|, |\partial\bar{w}^0/\partial x|, |\partial\bar{w}^0/\partial t|)$ such that $\int_{-\infty}^{\infty} N(x)dx$ converges; (3) $\lim_{|x|\to\infty} \bar{w}^0=0$ uniformly for all t; and (4) $\lim_{t\to 0} \bar{w}^0=0$ uniformly for all x. If also $N(x)>\max(|\partial^2\bar{w}^0/\partial x^2|)$, then for any $M>0$ p is continuous for $0\leq y\leq M$, $0\leq t$ except at x_i, 0, t; $\lim_{|x|\to\infty} \bar{w}=0$ uniformly for $0\leq t\leq T<\infty$, $0\leq y\leq M<\infty$; $\lim_{y\to\infty} \bar{w}=0$ uniformly for $0\leq t\leq T$; and the solution $\bar{w}, p$ is unique. He also shows that if both

$$\lim_{t\to\infty} \bar{w}^0=\bar{w}_0(x) \text{ and } \lim_{t\to\infty} \partial\bar{w}^0/\partial t=0$$

uniformly for all x, then

$$\lim_{t\to\infty} \bar{w}(x, y, t) \text{ and } \lim_{t\to\infty} p(x, y, t)$$

satisfy the steady state form of (*) and (**) with boundary values $\bar{w}_0(x)$. *J. H. Giese* (Aberdeen, Md.).

Finston, Morton. Free convection past a vertical plate. Z. Angew. Math. Phys. **7** (1956), 527–529.

The author shows that the boundary-layer equations for this problem can be reduced to ordinary differential equations when the ratio of the surface temperature of the plate to the free-stream temperature is of the form $T_s/T_\infty=1+Bx^\omega$, where x is the distance from the leading edge and B and ω are constants. *D. W. Dunn.*

★50 Jahre Grenzschichtforschung. Eine Festschrift in Originalbeiträgen. Herausgegeben von H. Görtler und W. Tollmien. Friedr. Vieweg & Sohn, Braunschweig, 1955. viii+499 pp.

The present volume is a collection of 44 original papers commemorating the fifty years of research in the boundary-layer theory since Prandtl's original paper on the subject [Verh. 3. Internat. Math.-Kongresses, Heidelberg, 1904, Teubner, Leipzig, 1905, pp. 484–491].

The papers included have not been published before in most cases, and appear to form a representative sample of present work in boundary-layer research. The historical development and basis of the theory are discussed only briefly, however, in the opening article by Tollmien. The general reader, although he will have to look elsewhere for a comprehensive treatment of the subject, will obtain from the book a useful survey of the problems of current interest. The research worker in boundary-layer theory will find many stimulating ideas for future investigation. *D. W. Dunn* (Ottawa, Ont.).

Geis, Theo. Bemerkung zu den "ähnlichen" instationären laminaren Grenzschichtströmungen. Z. Angew. Math. Mech. **36** (1956), 396–398.

In the study of similar solutions of the non-stationary boundary-layer equations, the transformations $u/u_1(x, t) = F'(\eta)$ and $\eta=y/N(x, t)$ lead to the differential equation

$$F'''+C_1(1-F')+C_2(FF''-F'^2+1)+C^0\eta F''+C^0FF''=0.$$

The coefficients C_i are in general functions of x and t, expressible in terms of the free-stream velocity $u_1(x, t)$ and the arbitrary function $N(x, t)$. The boundary conditions are $F(0)=F'(0)=0$ and $F'(\infty)=1$. The solutions of the boundary-value problem corresponding to constant coefficients C_i were considered by Schuh [see pp. 147–152 of the book reviewed above]. In the present paper, the author shows that there is another solution for non-constant coefficients C_i. *D. W. Dunn.*

Roberson, R. E.; and Tatistcheff, D. The potential energy of a small rigid body in the gravitational field of an oblate spheroid. J. Franklin Inst. **262** (1956), 209–214.

L'oggetto della Nota è il calcolo approssimato della energia potenziale gravitazionale mutua di due corpi di cui uno ha la forma di uno sferoide appiattito e l'altro un „piccolo" corps, tale cioè che i suoi punti hanno, rispetto ad un sistema di coordinate cartesiane aventi l'origine nel centro di massa, coordinate di cui si trascurano i quadrati e i prodotti (rispetto alle distanze dei punti stessi dai punti dello sferoide).

Alcune osservazioni finali si riferiscono a forme e strutture particolari dei due corpi. La ricerca mira al problema del moto di un satellite artificiale in prossimità della Terra. *G. Lampariello* (Roma).

Oswald, Telford W. A note on missile launching. J. Aero. Sci. **24** (1957), 74–76.

In a previous paper [same J. **23** (1956), 781–791; MR **18**, 348], the author has studied the problem of computing the launching trajectory of a missile which has small changes in inertia and stability margin. This note considers an approximate solution for the same problem when there are large changes in inertia and stability margin. The longitudinal motion is assumed to be uniform acceleration, the damping derivatives are assumed to be proportional to the velocity, and the pitching frequency is a particular function of the square of the speed. These assumptions check well with the actual values.

The resulting ordinary differential equation for the angle of attack can be solved in terms of Bessel functions, and the flight path angle and trajectory found by quadratures. Typical cases have been carried out and the results compare favorably with those found from step by step integration. *G. H. Handelman* (Troy, N.Y.).

Meksyn, D. Integration of the boundary-layer equations. Proc. Roy. Soc. London. Ser. A. **237** (1956), 543–559.

L'article est consacré à une intéressante résolution numérique des équations différentielles que l'on rencontre dans la théorie de la couche limite. La méthode est appliquée successivement à l'équation de Blasius, l'équation de Falkner et Skan, et l'équation relative au cas général d'un profil curviligne. Dans ce dernier cas, l'auteur montre que pratiquement, avec son procédé, l'équation aux dérivées partielles dont dépend le problème peut avec une bonne approximation être ramenée à une équation différentielle ordinaire. Les calculs numériques sont relativement simples. Dans les deux premiers cas, les résultats sont comparés aux résultats classiques d'Hartree; dans le dernier avec les résultats de mesures de Schubauer sur le cylindre elliptique. Ces comparaisons sont très favorables. La technique est essentiellement la suivante: Partant d'une solution obtenue par développement en série de puissance — évidemment non convergente dans l'intervalle d'étude $0<x<\infty$ — l'auteur porte le résultat dans l'équation non linéaire initiale pour la ramener à une équation linéaire, en général non homogène.

La résolution de cette dernière équation conduit à l'étude d'intégrales $\int_0^\infty \exp(-F(x))\phi(x)dx$, que l'on évalue par application de la méthode de descente (Debye)). E_n général la série obtenue n'est pas convergente; un procédé de sommation — lié au procédé classique d'Euler — conduit au résultat cherché. *P. Germain* (Paris).

Wood, W. W. Boundary layers whose streamlines are closed. J. Fluid Mech. **2** (1957), 77–87.

A class of two-dimensional boundary layers is considered for which the motion in the boundary layer can, in principle, be solved exactly. A closed solid boundary is supposed to move tangentially with uniform speed and the speed in the boundary layer is assumed to differ but slightly from that of the boundary. A formal solution may then be obtained by expansion of the velocity in a power series in a small parameter representation of the small differences of speed through the boundary layer. Examples are given of liquid within a fixed nearly circular cylinder which is lined by a moving band; and steady streaming of unbounded liquid past a rotating cylinder. Closed boundary layers in which the pressure is uniform, but the speed is not, are also considered.

L. M. Milne-Thomson (Providence, R.I.).

Morduchow, Morris. On laminar flow through a channel or tube with injection—Application of method of averages. Quart. Appl. Math. **14** (1957), 361–368.

By use of the method of averages, this paper obtains a simple approximate solution for the flow through a channel or circular tube with porous walls valid for all values of the injection velocity. The approximate solution is in each case made to satisfy an integral condition, rather than the exact equation of the problem. The solution reduces exactly to the small-perturbation solution for small values of the injection velocity, and gives a good approximation to the asymptotic solution for large values. In contrast to the effect for boundary layers, the skin friction coefficient for the channel is increased by fluid injection, except close to the entrance. For the circular tube, this is true at all distances from the entrance. *D. W. Dunn* (Ottawa, Ont.).

★ **Corcoran, Wm. H.; Opfell, J. B.; and Sage, B. H. Momentum transfer in fluids.** Academic Press Inc., New York, 1956. xi+394 pp. $9.00.

The stated purpose of this book is to present a unified account of some topics in fluid mechanics that are of special interest to chemical engineers, namely, laminar and turbulent shear flow, boundary-layer theory, and a statistical treatment of turbulence. The authors suggest that there is no single book available that treats these topics in a manner entirely suitable for their purposes. The presentation is on an elementary level, and is fairly conventional for the most part. The traditional topics are included, such as the mixing-length theory of turbulent shear flow. However, many of the concepts and results of the modern statistical theory of turbulence are also discussed, which is unusual in a book of this type. Long lists of references to original work are given, and in many cases the reader is referred to these for detailed mathematical analysis.

In the reviewer's opinion, many of the theoretical questions considered in this book are treated in a more satisfactory manner in previous books. The book will probably not be of too much general interest to mathematicians, but it does give a valuable insight into the engineering point of view.

D. W. Dunn (Ottawa, Canada).

Davies, V. C.; and Al-Arabi, M. Heat transfer between tubes and a fluid flowing through them with varying degrees of turbulence due to entrance conditions. Proc. Inst. Mech. Engrs. **169** (1955), 993–1000, discussion 1001–1006.

This paper describes an experimental investigation of turbulent heat transfer in a tube for different turbulence conditions at the commencement of heating. The results are correlated by the use of interpolation formulas involving the non-dimensional parameters of the problem.

D. W. Dunn (Ottawa, Ont.).

Haque, S. M. A. On the stability of a viscous liquid flowing in a straight circular tube under a constant pressure gradient. Pakistan J. Sci. Res. **6** (1954), 3–5.

The author claims to have proved the stability of the Poiseuille flow through a circular pipe by the method of positive definite integrals. The derivation appears to be in error even though the conclusion is generally accepted as plausible. *C. C. Lin* (Cambridge, Mass.).

Stuart, J. T. On the role of Reynolds stresses in stability theory. J. Aero. Sci. **23** (1956), 86–88.

The author clarifies the basis for the study of nonlinear oscillations in a channel against the criticisms of S. I. Pai. One of the main points is the fact that for a given rate of flow the pressure drop across a channel must necessarily be larger when there is a sustained oscillation of finite amplitude. *C. C. Lin* (Cambridge, Mass.).

Reid, W. H. On the transfer of energy in Burgers' model of turbulence. Appl. Sci. Res. A. **6** (1956), 85–91.

"The distribution of energy transfer is examined for two given energy spectra: one corresponding to an exponential correlation function and the other a block spectrum. Due to the absence of a pressure term, it is found that Burgers' equation does not lead to an algebraic behaviour of the correlation functions for large separations. It is also found that, in general, energy is transferred not only forward to higher wave numbers but also backwards to smaller wave numbers and, hence, that the usual cascade picture, in which energy flows in one direction only, may not be an exact one." (Author's summary.) *C. C. Lin* (Cambridge, Mass.).

Corrsin, S.; and Lumley, J. On the equation of motion for a particle in turbulent fluid. Appl. Sci. Res. A. **6** (1956), 114–116.

The authors discuss some considerations necessary in dealing with the problem stated in the title. A form for the equation is suggested. *C. C. Lin.*

Tatsumi, T. The theory of decay process of incompressible isotropic turbulence. Proc. Roy. Soc. London. Ser. A. **239** (1957), 16–45.

L'auteur étudie la turbulence isotrope d'un fluide incompressible. La densité spectrale d'énergie $\phi(k)$ est liée à la fonction $\varphi(k)$ qui représente en termes spectraux les corrélations triples par la relation classique

$$(1)\qquad \frac{\partial}{\partial t}\phi(k,t)+2\nu k^2\phi(k,t)=k\varphi(k,t).$$

A partir d'une relation introduite entre les distributions de probabilité, on retrouve tout de suite sur l'hypothèse souvent employée d'après laquelle les moments du 4ème ordre sont reliés aux moments du 2ème ordre comme si les lois de probabilité étaient normales. On aboutit ainsi à une seconde relation entre ψ et ϕ, qui prend la forme:

$$\psi=\psi_1+\psi_2 \tag{2}$$

avec

$$\psi_1(k,t)=2\pi\int_0^\infty\int_{-1}^1 \exp[-2\nu(k^2+k'^2+\mu kk')t] \times\psi(k,k',\mu;0)k'^2dk'd\mu,$$

$$\psi_2(k,t)=-4\pi\int_0^t dt'\int_0^\infty\int_{-1}^1 \exp[-2\nu(k^2+k'^2+\mu kk')(t-t')] \times[\phi(k,t')\{\phi(k',t')k+\phi(k'',t')\mu k'\} -\phi(k',t')\phi(k'',t')(k+\mu k')](1-\mu^2)k'^2k'd\mu,$$

$$\mu=\frac{\kappa\cdot\kappa'}{kk'},\quad \psi(k)=\int_0^\infty\int_{-1}^1\psi(k,k',\mu)k'^2d\mu,\ \kappa+\kappa'+\kappa''=0.$$

L'antisymétrie de $k\psi(k,k',\mu;0)$ entraîne que

$$\int_0^\infty k\psi(k,t)k^2dk=0.$$

L'ensemble des deux équations (1) et (2) peut en principe être résolu si l'on connaît $\phi(k,0)$ et $\psi(k,k',\mu;0)$. Cette dernière donnée est très mal connue, et l'auteur se contente dans la suite d'étudier la décroissance de la turbulence, telle qu'elle résulte des équations (1) et (2), et de données initiales choisies à priori à cause de leur simplicité analytique. Le problème peut alors être traité jusqu'au bout dans le cas limite d'un fluide non visqueux, où $\phi(k,t)$ est la seule fonction inconnue. Les cas examinés sont les suivants.

Spectre rectangulaire I: $\phi(k,0)=\begin{cases}\sigma_1 & (0\leq k<k_0)\\ 0 & (k_0\leq k)\end{cases}$

Spectre rectangulaire II: $\phi(k,0)=\begin{cases}0 & (0\leq k<k_1)\\ \sigma_2 & (k_1\leq k<k_0)\\ 0 & (k_0\leq k)\end{cases}$

Spectre uniforme: $\phi(k,0)=\sigma_1\ (k\geqq 0)$

Spectre à une seule raie: $\phi(k,0)=\sigma\delta(k-k_0)$.

Dans ce dernier cas les équations (1) et (2) se ramènent à une équation intégrale pour $\phi(k,t)$, dont on cherche les solutions développables suivant les puissances de ν et de t.

On donne les expressions des premiers termes du développement. On peut alors calculer approximativement la variation par rapport au temps de la fonction spectrale d'énergie et de l'énergie cinétique totale pour diverses valeurs du nombre de Reynolds. La solution trouvée n'a pas d'invariant de Loitsianski. Elle correspond assez bien à la première période de décroissance de la turbulence, malgré le caractère schématique de $\phi(k,0)$. *J. Bass.*

Meyer, R. E. On supersonic flow behind a curved shock. Quart. Appl. Math. **14** (1957), 433–436.

It is known that the shock-expansion theory gives accurate results for pressures along the body surface in plane supersonic flow over a pointed cylinder. On this basis it is proposed to make an approximate solution of the flow field by putting α=const along streamlines, where α is the characteristic parameter taken uniform downstream of the shock in the shock-expansion theory. The results show that under this new approximation the Mach lines are no longer isovels nor straight lines, as in shock-expansion theory, but are still isoclines and isobaed The theory leads to an integral equation which is solvrs. by iteration. *W. R. Sears* (Ithaca, N.Y.).

Germain, Paul. Ecoulements transsoniques avec onde de choc. C. R. Acad. Sci. Paris **243** (1956), 1190–1192.

F. I. Frankl' [Prikl. Mat. Meh. **19** (1955), 385–392; MR **17**, 550] has attempted to construct an approximate example of a plane flow which is subsonic above a certain streamline, while below it is supersonic up to a straight shock behind which the flow is again subsonic. His example is marred by the presence of an artificial discontinuity introduced to keep the velocity single valued. The author proposes a possibly more promising variation of the same general method for constructing a plane flow containing a locally supersonic region terminated at a shock. The equation $\eta\psi_{\theta\theta}+\psi_{\eta\eta}=0$, where ψ is the stream function, θ the inclination of the velocity, and η a function of the velocity, has solutions $\psi=(1+\lambda^3)^{im\theta^m}f(\lambda)$, where $\lambda^3=\eta^3\theta^2$ and f satisfies a hypergeometric equation. The simplified shock polar is defined by

$$(*)\qquad 2(\theta_2-\theta_1)^2+(\eta_2+\eta_1)(\eta_2-\eta_1)^2=0.$$

Then two curves $\lambda=\lambda_2>0$ and $\lambda=\lambda_1<0$ may be chosen to be images of the front and back sides of the shock if the correspondence between these curve is assumed to be $\theta_2=a^3\theta_1$, and then (*) defines $a=a(\lambda_1,\lambda_2)$. The shock conditions for ψ on these curves yield four conditions for a, m, λ_1, λ_2. If they are compatible, an unsettled question, one must also have in the region between the shock images $\psi=0$ on exactly two curves in $\eta\geqq 0$, and $\psi_\eta{}^2+\eta\psi_\theta{}^2>0$. The author shows that one must have $\lambda_1>-1$. He also shows that there are no suitable solutions for $a=1$, $-\lambda_1=\lambda_2<1$. Frankl's example corresponded to $a=1$, $-\lambda_1=\lambda_2=\infty$. *J. Giese.*

Dombrovskiĭ, G. A. An approximate solution of basic boundary problems for a plane supersonic steady potential motion of a gas. Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 799–802. (Russian)

The author has previously proposed [same Dokl. (N.S.) **103** (1955), 31–34; MR **17**, 681] a certain approximation to a coefficient in the system of partial differential equations for the potential and stream functions φ and ψ in a distorted hodograph plane. Then general solutions φ and ψ can be found explicitly in terms of two arbitrary functions, and the corresponding pressure-density relation can be adjusted to approximate $p/p_0=(\rho/\rho_0)^\gamma$ very well. In this paper he shows how to determine the arbitrary functions to satisfy the four following problems: φ and ψ prescribed (i) on a non-characteristic curve; (ii) on intersecting characteristics; (iii) on one characteristic, and $\psi=0$ on a free streamline; and (iv) on one characteristic, and $\psi=0$ on a prescribed profile in the physical plane. The first three can be solved by quadratures, at most. The last requires solution of a second order non-linear ordinary differential equation.

J. H. Giese (Aberdeen, Md.).

Holt, Maurice. The method of characteristics for steady supersonic rotational flow in three dimensions. J. Fluid Mech. **1** (1956), 409–423.

The method of characteristics for steady supersonic flow problems in three dimensions, due to Coburn and Dolph [Proc. Symposia Appl. Math., v. 1, Amer. Math. Soc., New York, 1949, pp. 55–66; MR **10**, 751], is extended so that flow with shocks and entropy changes

may be treated. Equations of motion based on Coburn & Dolph's characteristic coordinate system are derived and a scheme is described for solving these by finite differences.

A linearized method of characteristics is developed for calculating perturbations of a given three-dimensional field of flow. This is a generalization of the method evolved by Ferri [NACA Tech. Note no. **2515** (1951); MR **13**, 400] for perturbations of plane flow and conical flow. (Author's summary.) *W. J. Nemerever.*

Taylor, J. Lockwood. An exact solution of the spherical blast wave problem. Phil. Mag. (7) **46** (1955), 317–320.

An analytical solution is obtained of the equations governing the growth of a strong spherical explosion from a point source. The solution is a development of that due to Taylor [Proc. Roy. Soc. London. Ser. A. **201** (1950), 159–174, 175–186] for an explosion with constant total energy, and employs a similarity hypothesis. The exact solution is achieved by assuming that both kinetic energy and intrinsic energy are constant. The physical justification of this step is not discussed. *M. Holt.*

Cabannes, Henri. Sur les mouvements rectilignes non stationnaires d'un fluide compressible visqueux et conducteur. C. R. Acad. Sci. Paris **243** (1956), 1482–1484.

The problem of the unsteady, one-dimensional motion of a fluid is considered. The partial differential equations for the velocity, temperature and density are given in the general case when viscosity and thermal conductivity are non-zero. These equations are reduced to ordinary differential equations by considering only solutions independent of time. In this way it is shown that the velocity is in fact less, when viscosity and thermal conductivity are included, than when they are neglected. This result is to be expected from physical considerations.

G. N. Lance (Southampton).

v. Krzywoblocki, M. Z. On the optimum shape of an axially symmetric diffuser in compressible isentropic flow. Bull. Tech. Univ. Istanbul **9** (1956), 103–112. (Turkish summary)

The purpose of the note is to develop, by analytical means, a method by which the wall contour of an axially symmetric diffuser may be determined in terms of the radial and axial coordinates. The requirement which governs the construction of the optimum shape is that the pressure rise in the diffuser shall be maximum at the exit. The problem was attacked in the past by the author in two unpublished reports. The present approach represents a third attempt to solve the problem in an approximate way and is different from the previous approaches. (From the author's summary.)

J. B. Serrin (Minneapolis, Minn.).

Legendre, Robert. Séparation de l'écoulement laminaire tridimensionnel. Rech. Aéro. no. **54** (1956), 3–8.

Une ligne de séparation n'est ni une enveloppe de lignes de courant pariétales, ni un lieu de points de rebroussement de ces lignes. Elle est elle-même une ligne de courant pariétale et n'a aucune propriété locale. Sa seul caractéristique est de passer par un point de partage. Un point de partage est un point de l'obstacle par lequel passe une infinité de lignes de courant dirigées vers le fluide et s'étalant sur une surface de séparation. Cette surface coupe l'obstacle suivant la ligne de séparation. (Résumé de l'auteur.) *J. B. Serrin* (Minneapolis, Minn.).

Guiraud, Jean-Pierre. Sur la nature de la singularité d'un écoulement de fluide compressible au voisinage de la pointe avant d'une aile delta en régime subsonique. C. R. Acad. Sci. Paris **243** (1956), 2012–2014.

Cette note apporte une très intéressante précision à un résultat obtenu par le rapporteur [Rech. Aéro. no. **44** (1955), 3–8; MR **16**, 969] relatif au comportement d'un écoulement subsonique au voisinage de la pointe avant d'une aile delta; on montre en effet que localement, l'apex étant pris comme origine, le potentiel des vitesses est une fonction homogène des coordonnées, l'ordre d'homogénéité ν étant une fonction décroissante de la demi ouverture γ du delta, variant de 1 à $\frac{1}{2}$ lorsque γ croit de 0 à $\frac{1}{2}\pi$. La problème considéré et résolu par M. Guiraud est celui de la détermination de ν lorsque γ est petit. Il s'agit essentiellement d'une évaluation de valeur propre pour une équation intégrale dont le noyau s'exprime à l'aide de la fonction de Green. Pour γ petit, on met en évidence dans celle-ci la fonction de Green correspondant à $\gamma=0$. On peut alors étudier le problème de perturbation; on utilise à cet effet les fonctions propres du cas $\gamma=0$ qui sont des polynomes ultrasphériques de Gegenbauer. Le résultat est particulièrement simple: $\nu\equiv 1-\frac{1}{2}(1-\cos\gamma)$. *P. Germain* (Paris).

Ferri, Antonio; and Clarke, Joseph H. On the use of interfering flow fields for the reduction of drag at supersonic speeds. J. Aero. Sci. **24** (1957), 1–18.

This study is based upon the linearized theory of small-perturbation supersonic flows. The two general techniques of drag reduction by interference are described: one utilizes reflections and is typified by the famous "Busemann biplane"; the other attempts to produce neighboring wave patterns that cancel near the body as they propagate out into the fluid — this is the principle of the "area-rule" design, for example. Integral theorems are derived that give the total induction effects, over affected areas, due to source elements. These are used to calculate the interference drags of a wing near a reflection plate, certain three-dimensional biplanes, and a wing with a protuberance (thick portion near its root) and reflection plate. In the last case, for example, the plate can serve to eliminate most or all of the additional drag of the protuberance. The practicality of three-dimensional supersonic biplanes is discussed briefly. *W. R. Sears.*

Arsenin, V. Ya.; and Yanenko, N. N. On the interaction of shock waves with simple waves. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 713–716. (Russian)

In one-dimensional isothermal non-steady flow the region immediately behind a shock wave impinging upon a simple wave (in which one of the Riemann invariants is constant) is also a simple wave. The latter simple wave can be described by formulas involving quadratures of the functions that characterize the former simple wave. The author has applied his results to study a shock impinging upon a centered rarefaction wave. Two shocks arise, one of which advances through the simple wave, while the other recedes from it. The author shows that as time tends toward infinity the two shock velocities approach the values required for an instantaneous adjustment between the state ahead of the rarefaction wave and that behind the original shock. *J. Giese.*

Kompaneec, A. S. The similarity problem in the development of a shock wave from a compression wave. Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 29–32. (Russian)

The self similar, one dimensional, unsteady motion of a gas is considered, resulting from initial conditions

$$p_0=Bx^\alpha \ (x\geqq 0), \qquad \rho_0=\text{const} \qquad (\infty-\leqq x\leqq\infty),$$

$$p_0=0 \ (x<0), \qquad v_0=0$$

at time $t=0$. Here x denotes distance, p, ρ and v are the pressure, density and velocity respectively, and B and α are parameters. The physical variables may be written in the form

$$p=\rho_0\frac{x^2}{t^2}f(\xi), \quad v=\frac{x}{t}\varphi(\xi), \quad \rho=\rho_0\psi(\xi),$$

where

$$\xi=\left(\frac{B}{\rho_0}\right)^{1/(2-\alpha)}\frac{t^{2/(2-\alpha)}}{x}=\left(\frac{B}{\rho_0}\right)^{\beta}\frac{t^{2\beta}}{x}.$$

From the Eulerian equations of motion an ordinary differential equation of the first order may be derived to determine φ as a function of ω, where $\omega=f/\psi$. The integral curves of this equation are calculated and are seen to have singularities at the points $(0, 0)$ and $(0, 2\beta)$. *M. Holt.*

Lane, Frank. Supersonic flow past an oscillating cascade with supersonic leading-edge locus. J. Aero. Sci. **24** (1957), 65–66.

The Laplace-transform technique used by Miles in a problem of an oscillating airfoil in a wind tunnel [same J. **23** (1956), 671–678; MR **18**, 253] is used here to study staggered cascades in supersonic flow. The stagger is such that the locus of leading edges is not swept behind the Mach lines. Arbitrary phase lag between blades is permitted. The results reduce to Mile's for zero stagger and phase lag equal to π. *W. R. Sears.*

Matunobu, Yaso'o. Application of the thin-wing-expansion method to the compressible flow past an elliptic cylinder, with special reference to its fourth approximation. J. Phys. Soc. Japan **12** (1957), 72–84.

Both the complete and transonic thin-wing-expansion methods are utilized to obtain the steady irrotational flow of a compressible fluid past an elliptic cylinder. The velocity distribution over the surface is calculated up to the fourth powers in the thickness ratio t. Then it is found that the fourth order perturbations include not only terms of order t^4 but also those of order $t^4\log t$, and that the latter group becomes negatively predominant over the former in the transonic range. In calculating the fluid velocity near the stagnation point there remain insurmountable difficulties even for moderate subsonic flow. (From the author's summary.) *J. B. Serrin.*

Escande, Léopold. Remarque sur la stabilité des chambres d'équilibre ordinaires ou à montage Venturi. C. R. Acad. Sci. Paris **244** (1957), 148–149.

The stability of surge tanks used in hydro-electric power installations is examined using the linearized equation of Thoma. Analysis applies to only two types, the ordinary and the Venturi, but covers the cases of tank on either the inlet or outlet channels of the station. From the characteristic equation it is shown that same stability holds for both oscillating and aperiodic motions of the water surface in the tank. *W. D. Baines* (Ottawa, Ont.).

Lyon, Richard H. Response of strings to random noise fields. J. Acoust. Soc. Amer. **28** (1956), 391–398.

The response of a ribbon to a random distributed excitation is studied both theoretically and experimentally. With a turbulent flow along the ribbon, it is theoretically shown and experimentally verified, that there is a resonance phenomenon ("coincidence") when the flow speed is in approximate agreement with the wave speed. *C. C. Lin* (Cambridge, Mass.).

Levine, Harold. The torque on an infinite strip exposed to plane sound waves. Proc. Cambridge Philos. Soc. **53** (1957), 234–247.

Plane sound waves in two-dimensions are incident on a fixed rigid strip of width large compared to the wavelength. Starting with an approximation to the scattered field based on Sommerfeld's straight edge diffraction problem, an expression is developed for the mean torque on the strip. The torque is found to be oscillatory, a harmonic function of the ratio of strip width to wavelength. *W. W. Soroka* (Berkeley, Calif.).

See also: Hopf, p. 656; Diaz and Ludford, p. 655; Bendat, p. 679; Agostinelli, p. 699; Bertotti, p. 703; Bonnor, p. 705.

Optics, Electromagnetic Theory, Circuits

Stettler, R. Ueber radialsymmetrische optische Medien. Opt. Acta **3** (1956), 101–103.

The author considers optical media with central symmetry defined inside the unit circle in a plane, and the corresponding light rays, if the rays fulfill a certain boundary condition, namely that incidence angle and phase angle are proportional, the proportionality constant K being the same for all rays starting at the periphery of the unit circle gives, for different K the equations of the light rays. *M. Herzberger* (Rochester, N.Y.).

Croce, Paul. Etude d'une méthode de filtrage des images optiques. Rev. Opt. **35** (1956), 569–589, 642–656.

The author discusses the collaboration of optical image and photographic plate in the photographic image and suggests methods to minimize image deterioration, especially using frequency filters. He describes different such filters and their manufacture in detail, especially interference and polarizing filters and shows how they can be used to improve contrast in the photographic image. *M. Herzberger* (Rochester, N.Y.).

Satô, Yasuo; and Matumoto, Tosimatu. On the relay computer designed for correlogram analysis. Bull. Earthquake Res. Inst. Tokyo **34** (1956), 279–281 (1 plate). (Japanese summary)

The input is by teletype tape, which may be perforated either by keys or by a tracer, viz. a device which enters the readings on the tape as the operator traces over a record such as a seismogram. The computer calculates scalar products. *A. Blake* (Ann Arbor, Mich.).

Schiske, P. Die Behandlung der Übergangsstellen bei der Parabelmethode und ähnlichen Rechenverfahren. Optik **13** (1956), 529–536.

To compute the trajectories of electron optics one frequently uses an approximation, in which the axial path is approximated by successive parabolic arcs. These

paths can approximate the field in first order (Gaussian arcs), in third order (Seidel arcs) or to a higher order.

To determine the approximate path unequivocally, assumptions must be made with regard to the point where two consecutive arcs meet. This paper discusses several such assumptions, favoring a special one.

M. Herzberger (Rochester, N.Y.).

Kolomenskiĭ, A. A. On the theory of betatron oscillations of particles in magnetic systems. I, II. Ž. Tehn. Fiz. **26** (1956), 1969–1977, 1978–1990. (Russian)

The author studies the perturbed motion of a particle in the neighborhood of a given space curve under the action of a magnetic field. Stability and different types of linear and non-linear resonance phenomena are considered which may be important for the theory of particle accelerators. The precise results are very difficult to summarize because of their great number and complexity.

J. L. Massera (Montevideo).

Hoffman, John D.; and Axilrod, Benjamin M. Theory of dielectric relaxation for the three-dimensional polar rotator: lattice models leading to bimodal loss curves. J. Res. Nat. Bur. Standards **58** (1957), 61–73.

The characteristics of the dielectric relaxation spectrum associated with some simple three-dimensional lattices consisting of polar molecules of specified shape have been investigated with the object of determining whether dielectric loss curves with two distinct maxima (bimodal loss curves) could be predicted on a theoretical basis for such systems. For the types of lattice and molecular shape considered, each dipole has one stable and four metastable orientations. — The calculations show that under certain circumstances bimodal loss curves may arise for pear-shaped molecules in both the body-centered orthorhombic and tetragonal systems. All of the models lead to a single loss peak if the barrier system is sufficiently isotropic as in the body-centered cubic lattice. The changes with temperature of the shape of the loss curves, the static dielectric constant, and the configurational entropy are discussed. The abrupt changes in these properties that will occur at phase transitions due to cooperative interaction are also considered. Qualitative predictions concerning the expected behavior of bimodal loss curves for three-dimensional lattices, in which each dipole has only a single stable orientation, are compared with experiment. — Using an argument based on the fact that broadened loss curves in monophase molecular crystals become narrower with increasing temperature, it is concluded that fluctuations of structure are not the principal cause of the broadening of loss curves. The present theory leads to the correct type of temperature dependence, and this strengthens the view that the origin of multiple relaxation times in molecular crystals is the anisotropy of the crystalline field. *W. Nowacki* (Bern).

Manarini, Anna Marisa. Sulla propagazione delle onde piane nei conduttori in moto. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 229–231.

L'oggetto della Nota è di ritrovare alcuni risultati recenti dell'elettrodinamica di Minkowski dei corpi in moto, dovuti a Lampariello [gli stessi Rend. (8) **17** (1954), 37–44, 100–108, 222–228; MR **16**, 775, 1076; **17**, 216], Carini [Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **19(88)** (1955), 152–158; MR **17**, 561], Zeuli [Univ. e Politec. Torino. Rend. Sem. Mat. **14** (1954–55), 141–158; MR **18**, 357]nel caso molto particolare che sia utile l'uso diretto della trasformazione speciale di Lorentz, senza considerare le equazioni di Maxwell-Minkowski. Codesto metodo si ispira all'osservazione contenuta nelle recensioni del prof. J. L. Synge [MR **16**, 775, 1076, 1337] la quale tenderebbe ad attenuare l'importanza delle stesse equazioni di Minkowski. Ma è evidente che il riferimento a codeste equazioni e quindi per esse alla loro risolvente non puó, in generale, essere evitato quando intervengano nel problema condizioni al contorno riferentisi insieme a corpi in quiete e corpi in moto.

G. Lampariello (Roma).

Toraldo di Francia, Giuliano. Il metodo variazionale di Levine e Schwinger, applicato agli schermi a conduttività unidirezionale. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **21** (1956), 86–91.

The problem of the diffraction of electromagnetic waves by a plane screen having unidirectional conductivity may be reduced to the solution of an integro-differential equation of a complicated type. It is necessary therefore to consider methods of approximating to the solution.

In the present paper, it is shown that the integro-differential equation leads to a variational principle of the Schwinger-Levine type. *E. T. Copson.*

Fournet, G. Valeur limite, pour les très faibles angles, de l'intensité observable du rayonnement X diffusé par les fluides. J. Phys. Radium (8) **17** (1956), 940–943.

Bekannterweise erhält man die Röntgenstreuung eines isolierten Atoms auf dem Wege, dass man die Streuung eines freien Elektrons $I_e(s)$ mit dem Quadrat des sogenannten Atomformfaktors $F(s)$ (der die Ladungsverteilung und die demzufolge auftretenden Phasenunterschiede innerhalb des Atoms berücksichtigt) multipliziert. Werden in einen Gasvolumen N solche Atome bestrahlt, so muss man das erwähnte Produkt einfach mit N multiplizieren, weil alle Phasenunterschiede ungeordnet auftreten. Man hat also $I_0(s)=I_e(s)NF^2(s)$. Da jedoch die Verteilung auch schon wegen dem endlichen Volumen der Atome nicht ganz regellos sein kann, so tritt, besonders in Flüssigkeiten zum erwähnten Resultat noch ein Korrektionsfaktor hinzu, den man in den gewohnten Bezeichnungen

$$\left\{1+n_1\int_0^\infty [P(r,n)-1]\,\frac{\sin 2\pi rs}{2\pi rs}\,4\pi r^2 dr\right\}$$

schreibt. Der Verfasser bespricht die Frage, wie die Streuung im Falle von extrem kleinen Ablenkungswinkeln sein wird. In diesem Falle ist ein gewisser Übergang zu der Erscheinung verwirklicht, bei der keine Phasenunterschiede auftreten und die gestreute Intensität deshalb zu dem Quadrat der Zahl der streuenden Atome (N) proportional sein wird. Es tritt also auch noch ein solches Glied auf. (In einem gewissen Sinne ist das eine Art Mie-Effekt an materiell nicht mehr zusammenhängenden Teilchen.) Bezeichnen wir die zu N proportionale Streuintensität, die allein beobachtbar bzw. extrapolierbar ist, mit I_0 und die zu N^2 proportionale mit I_p, so folgt für die totale, die bei sehr kleinen Streuwinkeln tatsächlich auftritt $I_t=I_0+I_p$. Der Verfasser vergleicht seine Resultate mit die anderer Autoren und zeigt, dass bezüglich der beobachtbaren Intensität die sich auf diese Erscheinung beziehenden klassischen Formeln die richtigen sind.

T. Neugebauer (Budapest).

Kaprielian, Zohrab A. Electromagnetic transmission characteristics of a lattice of infinitely long conducting cylinders. J. Appl. Phys. **27** (1956), 1491–1502.

Im Mikrowellengebiet besitzen künstliche Medien, die z.B. aus Platten, Bänder, Stangen usw. aufgebaut sind und die die Eigenschaften der Dielektrika im optischen Gebiet nachahmen, als Linsen, Polarisatoren usw. eine grosse Bedeutung. Die vorliegende Arbeit beschäftigt sich deshalb mit den Eigenschaften eines aus äquidistanten unendlich langen leitenden Stäbchen aufgebauten Mediums. Bis jetzt wurden besonders zwei Methoden zur Berechnung der Eigenschaften solcher Media angewandt. Die molekulare Methode, die ein vollständiges Analogon der klassisch theoretischen Behandlung der Dielektrika im optischen Gebiet ist und die zellulare Methode, die das Analogon der Wigner-Seitzschen Methode in der Theorie der Kristallgitter ist. Der Verfasser arbeitet hier vier verschiedene Methoden aus. Erstens die Analogie des molekularen Aufbaues, die einfach zu einer zweidimensionalen Clausius-Mosottischen Gleichung führt, zweitens die Analogie der elektrischen Leitungsnetze, die schon die endliche Dicke der Sätbchen berücksichtigt, drittens eine Methode die auf der Summierung von den von den einzelnen Stäbchen gestreuten Wellen beruht (wobei das Resultat mit Hilfe von Besselschen und Hankelschen Funktionen ausgedrückt wird) und die bei alleiniger Berücksichtigung der induzierten Dipole in die erste übergeht und viertens eine formal ganz streng gültige Methode, die eine Anwendung des Variationsprinzips ist und auf das Problem der Lösung einer Integralgleichung führt. (Dieses Variationsprinzip wurde ursprünglich zur Lösung von wellenmechanischen Problemen eingeführt und dan auch auf elektromagnetische Probleme angewandt.) Die erhaltenen Resultate werden in Wellenleiter und im freien Raum mit der Erfahrung verglichen.

T. Neugebauer (Budapest).

De, J. On the equilibrium configuration of magnetic stars. Z. Astrophys. **40** (1956), 21–27.

A conducting fluid medium with no internal motions and with a prevalent magnetic field (**H**) is considered. If the magnetic field is not to decay then $\nabla^2\mathbf{H}=0$; and if the configuration is in addition to be in hydrostatic equilibrium curl $\mathbf{H}\times\mathbf{H}$ must be expressible as the gradient of a scalar function. If it is now supposed that the field is toroidal and axisymmetric then the only type of field which can be considered is $\mathbf{H}=H_\phi=H_0 r\sin\theta$, where H_0 is a constant and r, θ and ϕ are polar coordinates. On the boundary of a configuration a toroidal field must generally vanish. This condition cannot be met by the solution $H_0 r\sin\theta$. The author's further considerations relative to the possibility of imbedding such fields in slightly distorted fluid spheres would not appear to have much physical content. [It can be readily shown that toroidal axisymmetric fields which can prevail in a medium of infinite electrical conductivity and with no internal motions can depend only on $r\sin\theta$; this dependence is generally incompatible with the condition that it vanish on a given surface, e.g. the most general toroidal fields in a sphere are described by the modes

$$T_{n,j}=J_{n+3/2}(\alpha_{j,n}r)C_n^{3/2}(\mu)r^{-3/2},$$

where $\alpha_{j,n}$ is the jth zero of the Bessel function $J_{n+3/2}(x)$ and $C_n^{3/2}$ is the Gegenbauer polynomial [Chandrasekhar, Astrophys. J. **124** (1956), 244–265, eq. (36); MR **18**, 86].

S. Chandresakhar (Williams Bay, Wis.).

Agostinelli, Cataldo. Onde magneto idrodinamiche in una massa fluida incompressibile cilindrico circolare indefinita. Univ. e Politec. Torino. Rend. Sem. Mat. **15** (1955–56), 107–120.

In this paper the author studies the propagation, in the axial direction, of magneto-hydrodynamical waves of given frequency in a viscous incompressible electrically-conducting fluid contained in an infinite circular cylindrical tube under the action of a uniform magnetic field in the axial direction. In terms of cylindrical coordinates, (r, φ, z), so that the tube is $r=a$, say, the solutions sought are of the form

$$\mathbf{v}=\mathbf{v}(r,\varphi)e^{i(\alpha z+\omega t)},\quad \mathbf{H}=\mathbf{H}(r,\varphi)e^{i(\alpha z+\omega t)}$$

where ω is a given real constant. *E. T. Copson.*

Chopra, K. P. On the induction drag of a sphere moving in a conducting fluid in the presence of a magnetic field. Indian J. Phys. **30** (1956), 605–610.

A computation of the electromagnetic drag on a uniformly magnetized sphere moving through a medium of finite conductivity in translation parallel to the magnetic axis. The resistive moment to rotation about the axis is also calculated. Since polarization and induced field effects are neglected this would appear to have little application in fluid magnetics. *A. A. Blank.*

Stewartson, K. Motion of a sphere through a conducting fluid in the presence of a strong magnetic field. Proc. Cambridge Philos. Soc. **52** (1956), 301–316.

The steady motion of a perfectly conducting sphere in an inviscid electrically conducting fluid in the presence of a strong homogeneous impressed magnetic field **H** is considered. Two cases are distinguished: (1) When the sphere moves parallel to **H**, the ultimate flow and electromagnetic fields can be described as follows: A cylinder $\mathscr{C}$ of fluid is pushed along in front of the sphere and behaves as if the fluid were solid. Outside, the fluid moves parallel to the generators of $\mathscr{C}$ with a velocity which is singular at $\mathscr{C}$ and tends to zero as $r\to\infty$; this velocity is independent of z (the direction of **H**) and is given by

$$v_z=\frac{1}{\pi r}\left[3r^2\sin^{-1}\frac{1}{r}-\frac{3r^2-1}{(r^2-1)^{\frac{1}{2}}}\right]\text{for } r>1 \text{ and } v_z=1 \text{ for } r<1$$

where $r=(x^2+y^2)^{\frac{1}{2}}$ is measured in units of the radius of the sphere. Further there is (ultimately) no force on the sphere and the electromagnetic field is uniform; the initial perturbation in the magnetic field decays to zero as $t^{-\frac{1}{2}}$. (2) When the sphere moves parallel to the x-axis (say) and at right angles to **H**, the ultimate motion is again two-dimensional and is given by

$$u_z=0$$

$$u_r=\begin{cases}\cos\theta & \text{if } r<1,\\ \dfrac{\cos\theta}{\pi r}\left[3r^2\sin^{-1}\dfrac{1}{r}-\dfrac{3r^2-1}{(r^2-1)^{\frac{1}{2}}}\right] & \text{if } r>1,\end{cases}$$

$$u_\theta=-\frac{\tan\theta}{r}\frac{\partial}{\partial r}(ru_r).$$

In contrast to case (1) where the streamlines are straight lines parallel to the field and direction of motion, in case (2) they have to execute sharp turns.

S. Chandrasekhar (Williams Bay, Wis.).

Lock, R. C. The stability of the flow of an electrically conducting fluid between parallel planes under a transverse magnetic field. Proc. Roy. Soc. London. Ser. A. 233 (1955), 105–125.

It is known that for a plane-parallel flow (confined between $y=\pm a$) under a constant pressure gradient and in a transverse magnetic field (of strength H in the y-direction) the velocity profile is given by

$$(1) \qquad U=U_0\frac{\cosh M-\cosh(My/a)}{\cosh M-1},$$

where U_0 is the velocity at the center and $M=\mu Ha(\sigma/\rho\nu)^{\frac{1}{2}}$, μ, τ and ν are the coefficients of magnetic permeability, electrical conductivity and kinematic viscosity respectively, and ρ is the density. The stability of this flow is considered. With suitable simplifications, it is shown that the stability is governed by the Orr-Sommerfeld equation

$$(2) \quad (U-c)(u_y''-\alpha^2u_y)-U''u_y= -i(\alpha R)^{-1}(u_y^{(\mathrm{iv})}-2\alpha^2u_y''+\alpha^4u_y),$$

where $R=LU_0/\nu$ is the Reynolds number. In obtaining (2), the assumption has been made that the amplitude of the disturbance has the dependence $\exp[i\alpha(x-cU_0t)/L]$, where α is real and c can be complex. The noteworthy feature of equation (2) is that the magnetic field does not appear in this equation explicitly; it occurs only implicitly through the velocity profile (1). Equation (2) is discussed by an extension of Lin's method as developed for the hydrodynamic case. The principal result is that the transverse magnetic field has a very pronounced stabilizing effect. *S. Chandrasekhar.*

Bouwkamp, C. J.; and Casimir, H. B. G. On multipole expansions in the theory of electromagnetic radiation. Physica **20**, 539–554 (1954).

The authors derive the electromagnetic field from two scalar potentials. They show that the electromagnetic field in a region bounded by two concentric spheres is uniquely determined if the radial components of the electric and magnetic fields on those spheres are given. Spherical harmonic expansions for the radial components of fields and for the scalar (Debye) potentials are obtained. Properties of the modes thus introduced are discussed. *J. Shmoys* (Brooklyn, N.Y.).

Nisbet, A. Source representations for Debye's electromagnetic potentials. Physica **21** (1955), 799–802.

The author obtains some of the results of the paper reviewed above by a different method [cf. the paper reviewed below]. *J. Shmoys* (Brooklyn, N.Y.).

Nisbet, A. Hertzian electromagnetic potentials and associated gauge transformations. Proc. Roy. Soc. London. Ser. A. **231**, 250–263 (1955).

The author deals with the representation of the electromagnetic field in terms of electric and magnetic Hertz potentials. The wave equations satisfied by these potentials, when the physical sources of the field are prescribed, are obtained. A very thorough treatment of the gauge transformations on the Hertz potentials and on certain stream potentials (the latter appear in the inhomogeneous terms of the wave equations) is given. Some applications are given: treating the potentials in rectangular and spherical coordinates it is shown that with a suitable choice of gauge both potentials retain but a single component. *J. Shmoys* (Brooklyn, N.Y.).

Spătaru, Al. Détermination des conditions de travail en régime quasi-stationnaire et calcul des distorsions dans le cas de la modulation de fréquence. Acad. R. P. Romîne. Bul. Ști. Secț. Ști. Mat. Fiz. **8** (1956), 247–260. (Romanian. Russian and French summaries)

Pecher, Hans. Die Leitwertmatrix passiver und aktiver Netzwerke. Arch. Elek. Übertr. **10** (1956), 494–498.

The admittance (n-port) matrix of an electrical network is derived from the node representation, with arbitrary reference point; this is applied to passive and active (triode, p-n-p transistor) T-sections. The paper is tutorial. *H. G. Baerwald* (Cleveland, Ohio).

Neumann, Walter. Neue Fernleitungsnomogramme zum Lösen von Übertragungsaufgaben. Elektrotech. Z. **77** (1956), 900–903.

Henze, Ernst. Abriss einer verallgemeinerten Filtertheorie. Arch. Elek. Übertr. **10** (1956), 541–551.

A rigorous treatment of the theory of filtering based on the notion of an ideal filter introduced by the reviewer and K. S. Miller [J. Appl. Phys. **23** (1952), 223–228; MR **13**, 803]. Essentially, an ideal filter is interpreted as a projection in a function space of signal functions. The author considers various combinations of ideal filters and derives the conditions under which such combinations have the ideal filter property. He also shows how an analytical theory of such filters can be developed by introducing an appropriate basis in the function space, and illustrates the theory by several examples. *L. A. Zadeh* (New York, N.Y.).

Duffin, R. J. Exponential decay in nonlinear networks. Proc. Amer. Math. Soc. **7** (1956), 1094–1106.

The author first considers a differential equation of the type

$$dV_1(\dot{q})+R(q,\dot{q},t)\dot{q}+V_2(q)=0,$$

where $\dot{q}=dq/dt$; subject to the following conditions (i runs from 1 to 2): 1) $0=V_i(x)$; 2) there exist positive constants A, B such that

$$A\leq dV_i(x)/dx\leq B, \quad A\leq R(x,y,z)\leq B.$$

Under these conditions he shows that any solution of the differential equation decays exponentially. (This then generalizes a well-known result for constant coefficients.) The author then extends this theorem to several degrees of freedom, and in the process develops his theory of nonlinear definite transformations systematically and independently of his earlier papers on the subject. Finally, some physical corollaries are drawn from the general decay theorem: For instance, a network of linear transformers, linear capacitors, and quasi-linear resistors powered by batteries and alternating current generators cannot serve as power amplifiers. *R. Bott.*

Katz, Heinrich. Rechnende Netzwerke in der Regelungstechnik. Elektrotech. Z. **77** (1956), 560–565, 581–585.

An expository survey of the most common networks used in continuous automatic control circuits and of their basic functions: bridge circuits involving sum-difference formation for comparison and superposition of input-output and pre-set values in form of electrical quantities,

and simple R-C networks, partly in combination with additional feedback, as differentiatiors and integrators for enhancement of dynamic stability and rate of response as well as of precision of control. *H. G. Baerwald.*

See also: Macdonald and Brachman, p. 652; Harrison, p. 687; Guiraud, p. 696; Synge, p. 703; Mihul, p. 703.

Thermodynamics and Heat

Vodička, Václav. Circular cylinder under periodic fluctuations of temperature. Z. Angew. Math. Phys. **8** (1957), 53–64.

The paper is concerned with the determination of the temperature distribution in a circular cylinder of finite length when the boundaries of the cylinder are subjected to periodic changes of temperature.

In terms of cylindrical coordinates (ρ, φ, z), the problem is to find the solution $u(\rho, \varphi, z, t)$ of the equation

$$\frac{\partial u}{\partial t} = a^2 \nabla^2 u$$

in the region $0 \leqq \rho < r$, $0 \leqq \varphi \leqq 2\pi$, $-s < z < s$, such that

$$\frac{\partial u}{\partial \rho} + h_0[u - f(\varphi, z) \exp(i\omega_0 t)] = 0 \qquad (\rho = r),$$

$$\frac{\partial u}{\partial z} + h_1[u - g(\rho, \varphi) \exp(i\omega_1 t)] = 0 \qquad (z = s),$$

$$-\frac{\partial u}{\partial z} + h_2[u - h(\rho, \varphi) \exp(i\omega_2 t)] = 0 \qquad (z = -s),$$

where h_0, h_1, h_2 are constants, f, g, h given functions.

The solution will evidently, be of the form

$$u = u_0 + u_1 + u_2,$$

where u_0, u_1, u_2 are the three solutions in the three cases where only one of f, g, h is non-zero. These solutions are found by the usual method of separation of variables.

E. T. Copson (St. Andrews).

See also: Dutta, p. 686; De Groot and Mazur, p. 686; Tenerz, p. 687; Harrison, p. 687; Sharma, p. 689; Finston, p. 693; Davies and Al-Arabi, p. 694; Cabannes, p. 696.

Quantum Mechanics

Schwebel, S. L. Born approximation in a three-body scattering problem. Phys. Rev. (2) **103** (1956), 814–821.

The following problem is considered: Let $\psi(x_1, x_2)$ be the wave function for a system composed of two particles and a nucleus of infinite mass. Let H_0 be the Hamiltonian for the system if there were no interaction between the particles and let V be the potential of the interaction. Then $(H_0+V)\psi(x_1, x_2)=0$. Suppose initially the system is in a state in which particle 1 is incident with a given momentum K and particle 2 is bound; then

$$\psi(x_1, x_2) = \varphi_K(x_1)\varphi_0(x_2) + \psi_s(x_1, x_2),$$

where $\varphi_K(x_1)$ represents the state of the incident particle, $\varphi_0(x_2)$ the state of the bound particle, and $\psi_s(x_1, x_2)$ represents the scattering produced by the interaction potential. Note that $H_0\varphi_K(x_1)\varphi_0(x_2)=0$ and that $\psi_s(x_1, x_2)$ is a function satisfying the Sommerfeld radiation condition at infinity. Suppose $H_0=H_1+V_1$; then from

$$H_1(\varphi_K\varphi_0 + \psi_s) = -(V_1+V_2)\psi,$$

we find

$$(*) \qquad \psi = \varphi_K\varphi_0 + H_1^{-1}V_1\varphi_K\varphi_0 - H_1^{-1}(V_1+V_2)\psi.$$

By considering a specific example which can be solved exactly, the author shows that instead of using the standard Born approximation in which the function ψ on the right-hand side of (*) is replaced by the sum of the first two terms on the right-hand side, it is preferable to replace ψ by just the first term on the right-hand side.

B. Friedman (New York, N.Y.).

de Broglie, Louis; et Andrade e Silva, João. Idées nouvelles concernant les systèmes de corpuscules dans l'interprétation causale de la mécanique ondulatoire. C. R. Acad. Sci. Paris **244** (1957), 529–533.

Pour justifier, dans le cadre de l'interprétation causale de la mécanique ondulatoire par la théorie de la double solution, le succès de la mécanique ondulatoire des systèmes de corpuscules dans l'espace de configuration, les auteurs introduisent deux idées nouvelles qu'ils exposent dans le cas particulier d'un système isolé formé de deux corpuscules en interaction. [Author's abstract.]

The first of these ideas is to define the amplitude of the fictitious wave Ψ of configuration space as the product of the amplitudes of the two real waves of physical space. The second idea is to add a "quantum interaction" term to each of the propagation equations for these real waves; while this leads to the desired relation between quantum potentials, the specific form and meaning of these added terms is not yet apparent. *C. C. Torrance.*

Murty, G. S. On the relativistic Thomas-Fermi atom. Progr. Theoret. Phys. **15** (1956), 473–479.

The usual Thomas-Fermi equation for an atom confined in a finite volume is generalized to take into account the relativistic mass variation with velocity. It is shown that in the limit of high concentration, the distribution departs from an ideal Fermi-Dirac gas by terms of the order of the fine structure constant. The effect of the temperature term in the distribution function is also considered. *S. Chandrasekhar.*

Solov'ev, L. D. Infrared asymptotics of the electronic Green function computed with accuracy up to e^4. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 203–206. (Russian)

The theory of the renormalization group leads to an asymptotic expression in e^2 for the electrodynamic propagation function in the infrared region [N. N. Bogolyubov and D. V. Širkov, same Dokl. (N.S.) **103** (1955), 391–394: MR **17**, 441]. In this paper the method of perturbations is used to show that terms of order e^4 do not appear and therefore that the expression without change actually represents a higher-order approximation.

R. N. Goss (San Diego, Calif.).

Allcock, G. R. Normalization of Bethe-Salpeter wave functions. Phys. Rev. (2) **104** (1956), 1799–1802.

A formal theory of the normalization of the bound states of two particles as given by the Bethe-Salpeter equation is developed by use of the propagator, the Bethe-Salpeter component for one state and its adjoint for the other. *E. L. Hill* (Minneapolis, Minn.).

Wanders, G. Nonrelativistic limit of a Bethe-Salpeter equation. Phys. Rev. (2) **104** (1956), 1782–1783.

This is a preliminary note on an extension of the theory of the Bethe-Salpeter equation given by Wick and Cutkosky [Cutkosky, Phys. Res. (2) **96** (1954), 1135–1141; MR **16**, 656]. The "ladder approximation" is treated for two scalar particles which exchange scalar mesons of finite rest-mass. It is concluded that in the nonrelativistic limit for infintely heavy bound particles and zero binding energy, a subset of the spectrum for the Bethe-Salpeter equation is identical with that of the Schrödinger equation for a Yukawa potential. *E. L. Hill.*

Popovici, Andrei. Le déterminisme quantique et la loi de la réciprocité. II. Acad. R. P. Romîne. Bul. Sti. Sect. Sti. Mat. Fiz. **8** (1956), 121–181. (Romanian. Russian and French summaries)

Markham, Jordan J. Electron-nuclear wave functions in multiphonon processes. Phys. Rev. (2) **103** (1956), 588–597.

For most purposes, the wave functions of a crystal with an electron trapped at an impurity are approximated by the product of two functions, one involving the lattice coordinates and the other the position of the electron. The electronic function may be determined for the equilibrium position of the lattice (static approximation) or one may assume that the electronic wave function continuously adjusts itself to the instantaneous position of the lattice (adiabatic approximation). The author relates the wave functions to Hamiltonian operators which do not have some of the most common operational properties. A comparison of the two approximations is made using the variational principle. The static underestimates the kinetic energy and overestimates the potential energy, while the second does just the reverse. Although the formal treatment is quite general and includes all the effects for harmonic vibrations, the actual terms were evaluated for a simple model. The calculations show that for extremely shallow traps the static approximation may be slightly superior, while for deep traps the adiabatic approach should be used. For traps of depths less than 0.1 ev the methods are essentially equivalent; in any actual calculations, other approximations must be made and they are of greater importance than the slight difference between either of these approaches.

W. Nowacki (Bern).

Huang, Kerson; and Yang, C. N. Quantum-mechanical many-body problem with hard-sphere interaction. Phys. Rev. (2) **105** (1957), 767–775.

Roy, T. A note on the radiative corrections to Compton scattering. Proc. Nat. Inst. Sci. India. Part A. **22** (1956), 86–88.

The fourth order radiative correction to Compton scattering was calculated by Schafroth [Helv. Phys. Act. **22** (1949), 501–536] and by Brown and Feynman [Phys. Rev. (2) **85** (1951), 231–244]. These two calculations disagree. The present paper does not give a complete explanation of this discrepancy, but it shows that both of the above quoted papers took into account correctly the various contributions from the graphs, and so the source of the discrepancy must be in the actual evaluation of the integrals. *M. Moravcsik* (Upton, N.Y.).

Ganguly, S. Mixed nucleon-meson cascades in finite absorbers. Proc. Nat. Inst. Sci. India. Part A. **22** (1956), 40–53.

This paper is a contribution to the theory of cascades due to very high energy nucleon-nucleon collisions. It is an extension of work done by Messel, Potts, and McCusker [Phil. Mag. (7) **43** (1952), 889–899 inasmuch as in the present work the effect of the decay of the produced pions is also taken into account. It is claimed that this latter effect is quite decisive in determining the characteristics in showers in light elements, and so in air. The average number of mesons and nucleons produced by a primary of given energy at different depth is given. The method used is that of an analytic solution of the appropriate diffusion equations. It is unfortunate that the present paper contains no reference to some relevant Russian work by Landau. *M. J. Moravcsik* (Upton, N.Y.).

Gourdin, M. Diffusion nucléon-nucléon en présence de forces non centrales. I. Diffusion neutron-proton. J. Phys. Radium (8) **17** (1956), 988–996.

The differential cross section for neutron-proton scattering in the presence of tensor forces is expressed as an expansion in Legendre polynomials. Based on the method of partial waves in the $(-1)^L$ *SJM* scheme [F. Rohrlich and J. Eisenstein, Phys. Rev. (2) **75** (1949), 705–724] tables and formulae are given which suffice for the determination of the coefficients of the above expansion up to and including *F*-waves. The coefficients are functions of the phase shifts and of the amounts of admixture.

F. Rohrlich (Iowa City, Ia.).

MacDowell, Samuel Wallace. Polarization of spin one particles. An Acad. Brasil Ci. **28** (1956), 71–81=Notas de Fisica **2** (1955), no. 9, 20 pp.

The density matrix which characterizes the polarization of particles with spin [L. Wolfenstein, Annual Rev. Nuclear Sci. **6** (1956), 43–76] is expressed in terms of the expectation value of the spin vector and the symmetric second rank tensor associated with spin one particles. Its eigenvalues are determined and the possible degenerate cases are discussed. A geometrical interpretation is given in analogy with lightza polarition. *J. Leite Lopes.*

Bass, L. On the stochastic equation for the energy loss of fast electrons in matter. Proc. Indian Acad. Sci. Sect. A. **54** (1956), 423–427.

An electron of energy E moving through matter has probability $R(E'|E)dE'dt$ of going in the thickness dt to an energy between E' and $E'+dE'$, where R does not depend on t. For a given initial distribution of energy it is desired to find the energy density $\pi(E;t)$ at thickness t. The solution satisfies

$$\frac{\partial\pi(E,t)}{\partial t} = -\pi(E,t)\int_0^E R(E'|E)dE' + \int_E^{E_0} \pi(E';t)R(E|E')dE',$$

where $E_0 \leqq \infty$ is an upper bound for the initial energy. The author solves by reducing the equation to a Volterra-type integral equation, expressing the solution as a Liouville-Neumann type series. (The nth term of the series is the probability of energy E at t with exactly n emsisions between 0 and t, and the series can also be deduced directly from probabilistic considerations).

T. E. Harris (Santa Monica, Calif.).

See also: Raman, p. 686.

Relativity

Synge, J. L. Geometrical optics in moving dispersive media. Comm. Dublin Inst. Adv. Studies. Ser. A. no. **12** (1956), iii+63 pp.

The problem involved in the author's treatment is an attempt to treat a number of modern physical theories as the equivalent of geometrical optics in the space time of special relativity. The treatment is not completely general; the author makes certain plausible hypotheses with regard to the refractive index as a function of frequency.

The paper develops his ideas parallel to those of W. R. Hamilton with a terminology taken from wave mechanics working in Minkowski space, the geometry of which is discussed in some detail.

The motion of a photon, and the emission of a photon by an atom are some of the problems discussed.

M. Herzberger (Rochester, N.Y.).

Mihul, Eleonora. Une nouvelle formule pour l'interaction de deux charges en mouvement. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I. (N.S.) **1** (1955), 200–204. (Romanian. Russian and French summaries)

The author deduces the expression of the force of interaction — in Newtonian interpretation — between two spherical particles equally charged, using the Lagrange function, which corresponds to the metric of spherical symmetries of the corresponding system

$$ds^2=c^2\left(1-\frac{2V}{c^2}\right)dt^2-\frac{dr^2}{1-2Vc^{-2}}-r^2d\theta^2-r^2\sin^2\theta d\varphi^2,$$

where V is the classical potential of interaction.

The expression obtained depends on the relative velocity of these two bodies and represents, as well as the used interaction potential, a first approximation of the field.

O. Onicescu (Bucarest).

Marx, G. Innere Arbeit in der relativistischen Dynamik. Acta Phys. Acad. Sci. Hungar. **6** (1957), 353–379. (Russian summary)

The aim of this paper is the relativistic investigation of the types of forces different and more general then the Lorentz force. The starting points are the usual four dimensional laws of mechanics. It is shown that the action of such a general force changes the energy of the particle in two ways: either by changing its kinetic energy or by changing its rest mass. This later effect is carefully analyzed.

L. Infeld (Warsaw).

Bertotti, B. On gravitational motion. Nuovo Cimento (10) **4** (1956), 898–906.

The author develops an approximation method for solving Einstein's field equations (with energy-tensor) by expansion in powers of the gravitational constant. He shows that his method yields the usual results in the first step of the two-body problem (except for two terms which are non-linear in the masses and which one would not expect to find by this method). The method has the advantage that results appear in a Lorentz-invariant form.

Incidentally he finds the interesting result that if a world-tube of perfect fluid contains a streamline l on which the pressure is a maximum in each infinitesimal 3-space orthogonal to l, then the streamline is a geodesic.

F. A. E. Pirani (London).

Szekeres, G. Ether drift and gravitational motion. Phys. Rev. (2) **104** (1956), 1791–1798.

In a previous paper [Phys. Rev. (2) **97** (1955), 212–223; MR **16**, 869] the author defined an absolute time by a function τ such that τ=const is a space-like surface and whose gradient $\partial_\mu\tau$ is denoted by C_μ. He postulates that $g_{\mu\nu}$ and τ satisfy field equations $G_{\mu\nu}+T_{\mu\nu}+\frac{3}{4}\tau^{-2}g_{\mu\nu}=0$, where $G_{\mu\nu}$ is the Einstein tensor formed from $g_{\mu\nu}$ and $T_{\mu\nu}$ is a rather complicated combination of C_μ and $g_{\mu\nu}$ and their first derivatives.

The previous paper discussed a centrosymmetric particle at absolute rest. Here such a particle in absolute motion is discussed. By means of heroic calculations, using the method of Infeld and Schild [Rev. Mod. Phys. **21** (1949), 408–413; MR **11**, 816] the author shows that a free test particle with speed $\ll c$, travels on a geodesic. The field of the particle loses some of its symmetry due to ether drift effect. From available observational data the author concludes that for the sun the ether drift predicted by his theory is less than 100 km/sec.

A. J. Coleman.

García, Godofredo. The three "crucial effects" solved by the classical theory. Actas Acad. Ci. Lima **19** (1956), 3–26. (Spanish)

"In the present work we give a solution for the three 'crucial effects' by the classical Newtonian theory without recourse to Einstein's general theory of relativity."

From the introduction.

Trautman, A. On a generalisation of the Einstein-Infeld approximation method. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 439–442.

General formulas concerning the "new approximation method" [Einstein, Infeld and Hoffmann, Ann. of Math. (2) **39** (1938), 65–100] are given as the assumption of starting with a Galilean metric tensor is removed.

L. Infeld (Warsaw).

Trautman, A. Solution of one-body problem by the Einstein-Infeld approximation method. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 443–446.

The generalisation of the "new approximation method" developed in the paper reviewed above is used to find the gravitational field of a point mass. As a result the well-known Schwarzschild metric is obtained.

L. Infeld.

Costa de Beauregard, Olivier. Principe de Mach et univers en expansion. Un nouveau modèle de cosmos. C. R. Acad. Sci. Paris **244** (1957), 308–311.

The relation $\chi\rho R^2$=const, where χ is Newton's gravitational constant, ρ is the mean density of matter and R is the radius of the universe, is equivalent to assuming that the total mass of the universe is equal to its gravitational potential energy. This assumption, together with the relation $R=bt$, where b is a constant and t is time, are consequences of the assumption $3p+\rho=0$, where p is the pressure. Since $\rho>0$, this requires p to be negative, corresponding to continuous creation.

A. J. Coleman.

Loedel P., Enrique. A new principle which permits direct calculation of Einstein's gravitational potentials (g_{ik}). An. Soc. Ci. Argentina **160** (1955), 7–36. (Spanish)

The 'principle of parabolic velocity' is expressed by the author in the postulate: The behavior in a small region Q

of a static gravitational field, at rest with respect to an inertial system K, is the same as that in a system K' which moves with respect to K with the local parabolic velocity and in the direction of the gradient of the potential. This he applies, as in a previous article [Actas Acad. Ci. Lima **17** (1954), 3–38; MR **16**, 1165], to the spherically symmetric field, deriving thereby the Schwarzschild line element. He then applies the principle to the static field described by a general Newtonian potential $\Phi(x)$. This results in a static field $g_{00}=1+2\Phi$, $g_{ab}=g_{ab}(x)$, $(a, b=1, 2, 3)$, which rigorously satisfies the equation $R_{00}=0$ outside of matter, but not the spatial equations $R_{ab}=0$ — for stresses are required to keep the gravitational centers at fixed distances apart. The author finally applies his methods to the cases of system K' moving with constant acceleration and in hyperbolic motion with respect to an inertial system K, and concludes that the equivalence applies only within a differential neighborhood of space-time. *H. P. Robertson.*

Buchdahl, H. A. Reciprocal static solutions of the equations of the gravitational field. Austral. J. Phys. **9** (1956), 13–18.

The author extends the results of a previous paper [Quart. J. Math. Oxford Ser. (2) **5** (1954), 116–119; MR **16**, 531] to line elements that remain valid inside matter. Applications are made to fluid distributions and electrostatic fields. *M. Wyman* (Edmonton, Alta.).

Takeno, Hyôitirô. On the theory of gravitational waves. Tensor (N.S.) **6** (1956), 15–25.

The problem of gravitational waves has been studied in general relativity by means of the linearized equations. However the properties of an approximate solution are not necessarily the same as those of the corresponding rigorous solution. For this reason the author treats the same problem from a rigorous point of view.

The gravitational potentials are assumed to be functions of a single variable and suitable coordinate conditions are introduced. The corresponding solutions of the field equations and some of their physical implications are compared with the results obtained from the linearized equations. *M. Wyman* (Edmonton, Alta.).

Narlikar, V. V. Results of gravitational significance in Riemannian geometry. Nature **177** (1956), 1138.

The author gives an interpretation to the cosmological constant in terms of an arbitrary vector field.
M. Wyman (Edmonton, Alta.).

Bass, L.; and Pirani, F. A. E. On the gravitational effects of distant rotating masses. Phil. Mag. (7) **46** (1955), 850–856.

In Thirring's calculation of the gravitational field near the center of a rotating spherical shell the energy-momentum tensor is taken to have the form

$$T^{ij}=\rho v^i v^j, \tag{1}$$

where ρ is the density and v^i the velocity of matter composing the shell. However to the degree of approximation used (1) does not satisfy the usual conservation laws. To rectify this error the authors introduce an additional term E^{ij} which is interpreted as representing the elastic stresses.

From simple physical assumptions the form of E^{ij} is determined and the corresponding approximate equations of motion are determined. *M. Wyman.*

Mishra, R. S. Basic principles of unified field theory. Nuovo Cimento (10) **4** (1956), 907–916.

A brief comparison is made of the field equations used in the unified field theories developed by Einstein, Schrödinger and Bonnor. *M. Wyman.*

Ghosh, N. N. On a solution of field equations in Einstein's unified field theory. I. Progr. Theoret. Phys. **16** (1956), 421–428.

By taking a specialized form for the gravitational potentials the author discusses rigorous solutions of the strong form of Einstein's unified field equations.
M. Wyman (Edmonton, Alta.).

Husain, S. I. Conservation laws and other identities in Bonnor's unified field theory. Nuovo Cimento (10) **4** (1956), 768–778.

The author shows that the conservation laws that hold in the Schrödinger and Einstein unified field theories also hold in the form given by Bonnor. *M. Wyman.*

Kichenassamy, S. Sur les conditions d'existence de la solution de $g_{\mu\nu;\rho}=0$. C. R. Acad. Sci. Paris **244** (1957), 168–170.

The Einstein equations

$$\partial_\omega g_{\lambda\mu}-\Gamma_{\lambda\omega}{}^\alpha g_{\alpha\mu}-\Gamma_{\omega\mu}{}^\alpha g_{\lambda\alpha}=0 \tag{1}$$

admit a unique solution for $n=4$ if and only if

$$g\{[2(1+3k)-g]^2-4k[3+k-g]^2\}\neq 0, \tag{2}$$

where $g=\det((g_{\lambda\mu}))/\det((g_{\omega\mu}))$, $k=\det((g_{\lambda\mu}))/\det((g_{\omega\mu}))$ [M. A. Tonnelat, same C. R. **230** (1950), 182–184; J. Phys. Radium (8) **12** (1951), 81–88; **16** (1955), 21–38; MR **11**, 569; **13**, 79; **16**, 755; Hlavatý, J. Rational Mech. Anal. **2** (1953), 1–52; **4** (1955), 247–277; Hlavatý and Sáenz, ibid. **2** (1953), 523–536; MR **14**, 505; **16**, 870; **14**, 1132]. Tonnelat's method for solving (1) is less restrictive, for she admits also the possibility $k>0$, while Hlavatý confines himself for $n=4$ to the physically admissible cases $k\leqq 0$. The author points out this difference.

{Reviewer's remarks: 1) Hlavatý's method for $n=4$ quoted by the author of the paper under review is a special case of a method for general n, where the sign of k is irrelevant [ibid. **2** (1953), 1–52]. 2) Author's statement about $\lambda_{ijk}\neq 0$ is not correct for the second class, because for $g=2$ we have $\lambda_{223}=\lambda_{114}=\lambda_{113}=\lambda_{224}=0$. 3) The solution of (1) for the case when (2) is not satisfied was given explicitely by Hlavatý [ibid. **4** (1955), 653–679; MR **17**, 200].} *V. Hlavatý* (Bloomington, Ind.).

Horváth, J. I. Contribution to Stephenson-Kilmister's unified theory of gravitation and electromagnetism. Nuovo Cimento (10) **4** (1956), 571–576.

A geometrical interpretation of Stephenson-Kilmister's unified theory is attempted on the base of Finslerian geometry. *L. Infeld* (Warsaw).

Pham Tan Hoang. Conditions de conservation pour le tenseur d'impulsion-énergie en théorie unitaire d'Einstein-Schrödinger. C. R. Acad. Sci. Paris **243** (1956), 1600–1603.

In unified field theories based on asymmetric g_{mn}, the metric tensor a_{mn} is often identified with

$$h_{mn}=\tfrac{1}{2}(g_{mn}+g_{nm}).$$

The author shows that identifications of a_{mn} with $(h/g)^{\frac{1}{2}}h_{mn}$ permits the definition of a symmetric diver-

genceless tensor T_{mn} which he proposes should serve as the energy-momentum-density tensor. *A. J. Coleman.*

Astronomy

Agekyan, T. A. On the theory of fluctuations in brightness of the Milky Way and of the Metagalaxy. Vestnik Leningrad. Univ. **11** (1956), no. 1, 145–154. (Russian)

The paper consists of two parts: In the first part the differential equations derived by Chandrasekhar and Münch [Astrophys. J. **112** (1950), 380–392; MR **12**, 644] for describing the fluctuations in brightness of the Milky Way are generalized to the case when the emission per unit volume and the probability of occurrence of a cloud per unit length along the line of sight are both functions of position. Explicit expressions for the moments of the relevant distribution functions are given. In the second part of the paper the correlation in brightness in two directions inclined at a small angle is considered. The author obtains a linear homogeneous partial differential equation for the required correlation coefficient when considered as a function of the extent of the system along the line of sight and of the angular separation. The formal solution of this equation is written down and various special cases are considered. *S. Chandrasekhar.*

Kuščer, I. Milne's problem for anisotropic scattering. J. Math. Phys. **34** (1956), 256–266.

The asymptotic (at great depths) of the solution of the equation of radiative transfer appropriate for a semi-infinite plane-parallel atmosphere scattering in accordance with a general phase function is considered. It is shown that the required asymptotic solution can be expressed uniquely in terms of a suitably defined H-function. [For definition see Chandrasekhar, Radiative transfer, Oxford, 1950, chap. v; MR **13**, 136.] The author's analysis clarifies certain questions relating to the so-called "pseudo-problems" in transfer theory to which Chandrasekhar has drawn attention [loc. cit., p. 89].

S. Chandrasekhar (Williams Bay, Wis.).

Bonnor, W. B. Boyle's law and gravitational instability. Monthly Not. Roy. Astr. Soc. **116** (1956), 351–359 (1957).

A spherical mass of gas of uniform chemical composition is in isothermal equilibrium under its own gravitation. At every point the pressure (p), density (ρ), temperature (T) and molecular mass (m) are related by $p=kpT/m$, where k is the gas constant. If N is the total number of molecules, V the volume within radius r and p the boundary pressure, it is shown that

$$pV=NkT-\frac{16}{3}\pi^2G\int_0^r \rho y\,dy\int_0^y \rho x^2\,dx,$$

whereas Boyle's Law would give merely $pV=NkT$. The $p-V$ curves are constructed for a gas inside a spherical container of variable radius which holds a fixed number of molecules and is kept at constant temperature, the external pressure being varied and the gas, after each variation, being supposed to come to isothermal equilibrium. If the gas is sufficiently compressed, its equilibrium is unstable. For a fixed mass of gas, stable configurations can occur with radii greater than a certain value, which is calculated, up to values for which V is infinite. But it is also shown that any sufficiently large isothermal sphere is unstable and the criteria for this are compared with Jeans' criterion of instability. The latter implies that the unperturbed density is constant, a feature that can be criticised for a gravitational distribution of matter. The relevance to the formation of condensations in an expanding universe that began from a point-source concentration of gaseous material is discussed. It seems unlikely that condensations comparable with the galaxies could form in such a universe if its age did not exceed 4×10^9 years. *G. C. McVittie.*

See also: Counson, Ledoux et Simon, p. 686; De, p. 699.

Geophysics

Bellman, Richard; and Kalaba, Robert. On the principle of invariant imbedding and propagation through inhomogeneous media. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 629–632.

The authors obtain an integral equation for the scattering function describing diffuse reflection by a plane-parallel atmosphere in which the albedo for single scattering is a function of position. (The problem considered by the authors has been treated under somewhat less general conditions by the reviewer [Radiative Transfer, Oxford, 1950, ch. vii, 51; MR **13**, 136]; the authors have overlooked three other equations which govern the problem.) *S. Chandrasekhar.*

Kasahara, Keichi. Strain energy in the visco-elastic crust. Bull. Earthquake Res. Inst. Tokyo **34** (1956), 157–165. (Japanese summary)

Matuzawa's theory of the transfer from heat to strain energy in the earth's crust is here modified to allow for dissipation of part of the energy by viscous flow. The author notes a possible correlation between the viscosity of the rock and the periodicity of earthquakes.

A. Blake (Ann Arbor, Mich.).

See also: Gaskill, p. 675; Bendat, p. 679.

OTHER APPLICATIONS

Games, Economics

Isbell, J. R.; and Marlow, W. H. Attrition games. Naval Res. Logist. Quart. **3** (1956), 71–94.

This paper is a summary report including summaries of studies preliminary to the work described here. The games discussed are stochastic games based on the Lanchester theory of attrition. The Lanchester theory [Aircraft in warfare, the dawn of the fourth Arm, Constable, London, 1916; also Morse and Kimball, Methods of operations research, Wiley, New York, 1951; MR **13**, 364] describes in probabalistic terms the attrition rates of opposing military forces committed to combat. The process becomes a stochastic game [see L. S. Shapley, Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 1095–1100; MR **15**, 887] when the possibilities for command decisions at various stages are inserted. The authors discuss Hotspot games [see C. Tompkins, George Washington Univ. Logist. Papers no. **2** (1950)] in which the commanders commit various forces from reserve to combat at a contested point

at various times, and programming games in which each commander may deploy his weapons of different types in various ways against those of the opposing commander. And finally the question of weapon evaluation is taken up. Essentially, the ratio of the values of two weapon types is taken to be inversely proportional to the ratio of the numbers of weapons of each type that would result in a "stand-off". The authors take up the possibility of estimating the values directly from the parameters of the process. The results of many examples are included throughout the paper and the authors report many subjective considerations based on computations.

E. D. Nering (Tucson, Ariz.).

★ **Patinkin, Don. Money, interest, and prices. An integration of monetary and value theory, with a mathematical appendix.** Row, Peterson and Co., Evanston, Ill.-White Plains, New York, 1956. xix+510 pp.

This work represents a major step in reformulating monetary theory so as to clarify the problems and bring them into line with the methods of ordinary value theory. The implications of a consistent formulation in which choices as to money balances are made simultaneously with choices of commodities are drawn for the quantity theory of money and other aspects of the determination of the rate of interest and the absolute price level.

Chapter I sketches quickly a barter exchange economy; there are no novelties, but the exposition is unusually clear. Great stress is laid on the difference between individual- and market-experiments; in the first, the reactions of an individual to variables, such as prices, which are parametric to him are studied; in the second, the reaction of market equilibrium values of prices and other economic magnitudes to changes in the parameters of the system as a whole (such as the total quantity of money) are dealt with. The next four chapters extend the model to include money and bonds. Basically, it is assumed that there is an exogenous flow of commodity supplies to individuals now and over future weeks, that the utility function of each individual depends upon consumption in present and future weeks and upon the real value of the stock of money balances in the present and future, and that for each time period choices are made subject to a budget restraint which equates the cost of consumption and bond purchases plus desired money holdings to the value of commodity supplies plus repayments of previous bond holdings with interest plus initial money holdings. Future prices are expected to be the same as current ones. (In this model, bonds do not directly enter the utility function, though they have derived value because of interest payments.) This is an economy without money illusion in the sense that doubling of all prices and of all initial money holdings leaves the real situation completely unaltered. Market equilibrium is defined by equality of supply and demand on all markets (commodities, bonds, and money, but by Walras' Law, equality on the last follows from equality on all the others) for the current period; future periods only enter in the individuals' plans but have no market implications. It is then shown that an equi-proportionate change in initial money holdings will leave the equilibrium rate of interest unchanged and change all commodity prices proportionately. Thus the quantity theory of money derives solely from the absence of money illusion and is not tied up with various special assumptions given in the literature. A number of specific conclusions are drawn about the shape of the demand function for money. {But considerably more use of the apparatus of demand analysis could have been made.}

The most important implication drawn from this analysis is an attack on the frequent dichotomization of the pricing process, in which, it is held, the relative prices of commodities are determined by the equilibrium conditions in those markets, and the absolute price level determined by a separate and independent monetary equation. This model is shown to be logically contradictory, especially when viewed from the dynamic point of view, in which the approach to equilibrium is considered. Here the model is based on the Law of Supply and Demand, in that the price on each market is assumed to change proportionately to the difference between demand and supply. It is shown that the dynamics of the adjustment to an increase in the supply of money only makes sense if the demand and supply of commodities and bonds depend on the real value of the stock of cash balances as well as on relative prices.

Chapter VII is a digression in which an attempt is made to explain the attachment of utility to the holding of money balances. A model is suggested in which market transactions take place periodically, but between transaction times payments are made at random intervals. Then the amount of defaults equals the maximum excess during the time period of the cumulated payments owed over the cumulated payments received less the initial money holdings. The probabilities involved are computed in an Appendix by A. Dvoretzky. The utility of money is then its contribution to reducing the probability of default. {The author seems to imply that this explanation differs from that attributing the utility of money to imperfect facilities for credit, but it appears to the reviewer still to rest on the impossibility of borrowing money between market periods, which is an imperfection in the market. Randomness in the payments, while undoubtedly adding to realism, is not, as far as the reviewer can see, logically necessary to explaining the utility of money, once it is granted that default is not infinitely costly, since even under certainty of payments, an individual may prefer defaulting to holding too large an initial stock of money.}

In Chapters IX–XI a macroeconomic version of the model is presented. The properties of the various supply and demand functions are brought out, and the stability of the system demonstrated. A number of implications with respect to specific shifts in the functions are developed. The implications of dropping some of the assumptions are explored in Chapter XII. In Chapter XIII, the behavior of the model under conditions of unemployment (i.e., in disequilibrium or with rigid prices) is examined. A number of interesting problems as to the significance of the usual demand and supply functions under these circumstances are raised, though not resolved.

The macroeconomic model is then used to develop a Keynesian theory of unemployment as a dynamic phenomenon (rather than one involving unemployment equilibrium). Finally the whole development is used to criticize Keynesian interest theory and its relation to the neoclassical views.

The text develops its arguments verbally and graphically; mathematics is put into the appendices. The author even states that the latter are not necessary for the development of the text. {While it is true that at many points the verbal and graphical reasoning is sufficient, this is by no means always so; thus the graphical argument

for the stability of the macroeconomic system is in no way sufficient and is not equivalent to the (correct) proof in the Appendix.} The book is greatly concerned with the interpretation of earlier writers and concludes with a detailed and sparkling set of historical notes, which constitute a major contribution to the history of economic thought. *K. J. Arrow* (Stanford, Calif.).

Burger, E. On the stability of certain economic systems. Econometrica **24** (1956), 488–493.

Consider the difference-differential equation

$$\dot{y}(t)=ay(t)+by(t-\theta).$$

Then all roots λ of the corresponding characteristic equation $\lambda=a+be^{-\lambda}$ have negative real parts if and only if (1) in the case $b\geqq -1$ then $a<-b$, (2) in the case $b<-1$ then $a<b$, or $b\leqq a<-b$ and $\arccos -a/b>(b^2-a^2)^{\frac{1}{2}}$, where $0<\arccos -a/b\leqq\pi$. This is shown to be equivalent to a result of N. D. Hayes [J. London Math. Soc. **25** (1950), 226–232; MR **12**, 106]. An application to an economic model due to Föhl is discussed. *H. S. Houthakker.*

Banerjee, K. S. A comment on the construction of price index numbers. Appl. Statist. **5** (1956), 207–210.

Ruderman [Appl. Statist. **3** (1954), 44–47] was led to consider the average price, obtained from family budgets, of a group of commodities. The author studies the average obtained by weighting the prices of a sample of varieties with the amounts purchased in the market on a certain day. He shows that if the number of transactions for each variety is the same this average will give an unbiased estimate of the corresponding average price obtained from family budgets. *L. Törnqvist* (Helsinki).

★**Burkhardt, Felix. Anwendungen der Wahrscheinlichkeitsrechnung und der mathematischen Statistik in der Wirtschaft.** Bericht über die Tagung Wahrscheinlichkeitsrechnung und mathematische Statistik in Berlin, Oktober 1954,, pp. 65–72. Deutscher Verlag der Wissenschaften, Berlin, 1956.

An elementary review of acceptance sampling and of input-output analysis is followed by a briefly sketched model of economic circulation in which a single source distributes goods continuously to n other entities with probabilities varying in time. No applications of the model are given, and the principal formula (6) is incorrect. *K. J. Arrow.*

★**Abruzzi, Adam. Work, workers and work measurement.** Columbia University Press, New York, 1956. xvi+318 pp. $7.50.

This book attempts to broaden the basis for the theory and practice of work measurement; it emphasizes social and other values, and it delimits the role of purely statistical methods. A sharp distinction is made between an estimating function and an evaluation function. After a brief discussion of the design of an investigation, the author points out the difficulties of standardization, and insists that the measurement of time for production operations must be kept separate from the determination of production norms and wage rates. Shortcomings of conventional rating procedures are enumerated, a special point being the common confusion between precision and accuracy. A proposal is made to deal with the rating problem by game theory. A detailed analysis is made of time-measuring instruments and their operation. To achieve standardization, criteria are given for local and grand stability; these criteria are applied to a large number of practical situations. Delays are analyzed. The problems and failures of standard data systems, standard elements, and standard motions are described. In his concluding chapters, the author insists that "there simply is no 'one best way' in terms of all the requirements that must be considered in any real-life problem." He justifies this maxim by an analysis of purposive behavior at various levels. He discusses various forms of fatigue, the two kinds of skill, and their relation to work.

C. C. Torrance (Monterey, Calif.).

Dunn, Paul F.; Flagle, Charles D.; and Hicks, Philip A. The Queuiac: An electromechanical analog for the simulation of waiting-line problems. Operations Res. **4** (1956), 648–662 (1957).

Luchak, George. The solution of the single-channel queuing equations characterized by a time-dependent Poisson-distributed arrival rate and a general class of holding times. Operations Res. **4** (1956), 711–732 (1957).

The class of holding (service) times considered is that for which the distribution is a linear sum of gamma-type distributions; this is the same as saying that the holding time may be represented by a collection of phases of service, each with an exponential distribution. For such holding times, as is well-known, the differential recurrence state equations remain linear, the state being taken as the number of phases. With the arrival rate a power-series in the time variable, the author determines recurrence relations for certain coefficients related to the state transition probabilities in a similar power series. More explicit results are obtained in the two simpler cases: (i) constant arrival rate and a holding time certain to involve k phases, and (ii) linear arrival rate and exponential distribution of holding time. Those for the former are written in terms of a function resembling the Bessel I function, namely

$$I_n^k(z)=\sum_{r=0}^{\infty}(z|2)^{n+(k+1)r}/r!(n+rk)!$$

and for $k=1$ reduce to a result given by Ledermann and Reuter [Philos. Trans. Roy. Soc. London. Ser. A. **246** (1954), 321–369; MR **15**, 625]. Those for the latter are more complicated, a circumstance consistent with the result for general arrival time-dependence (and exponential service time distribution) obtained by A. B. Clarke [Ann. Math. Statist. **27** (1956), 452–459; MR **18**, 157].

{No discussion is given of the question of determining the arrival time-dependence from traffic measurements, consideration of which has led C. Palm [Ericsson Technics no. **44** (1943); MR **6**, 160] to a different kind of development.} *J. Riordan* (New York, N.Y.).

Burke, Paul J. The output of a queuing system. Operations Res. **4** (1956), 699–704 (1957).

An unsaturated s-counter queue with Poisson arrivals of density λ, and exponential service time is considered in the steady state. A rather remarkable observation is made: The departure epochs from the queue (that is, the instants at which some service ceases) are also a Poisson process (of density λ). This result has important applications in problems of queues in tandem. *E. Reich.*

Kesten, H.; and Runnenburg, J. Th. Priority in waiting line problems. Math. Centrum Amsterdam. Statist. Afdeling Rep. S 211 (VP 11) (1956), 26 pp.

The authors obtain extensive results on the queuing problem of A. Cobham [J. Operations Res. Soc. Amer. 2 (1954), 70–76]: One counter serves r types of customers; service time for a customer of type k has the distribution F_k, and all service times are independent; arrival times for customers of type k have independent Poisson distributions P_k; if two customers are ready to be served, the one having lower type is served first, but service of a customer is never interrupted. In the case of nonsaturation (expected new demands on service time during time dt less than dt), the authors show that the distribution of waiting times for customers of type k tends to a limit which is independent of the queue present when the counter opens, and give formulas for the first and second moments and Laplace transform of the stationary-state distributions in terms of F_k and P_k. *P. Wolfe.*

Haycocks, H. W.; and Plymen, J. Investment policy and index numbers. J. Inst. Actuar. 82 (1956), 333–371; discussion 372–390.

The aim of the authors is to review some of the developments that have taken place in the principles and practice of investment policy over the last twenty five years, and to study the contribution to these principles that the index numbers can provide. For this purpose the Actuaries' Investment Index is investigated. In the article the methods used in the construction and maintenance of the index are described in the light of current conditions. The method of calculation and usefulness of this index is compared with competing indexes. Some interesting remarks about the properties of the geometric mean used for the construction of security price index numbers are made. Further, the yields of securities are investigated. This part contains both theoretical results and results of an investigation into data taken from the working sheets of the Actuaries' Investment Index. It compares the results of investments in low-, medium-, and high-yielding ordinary shares. The low-yielding group has been the most profitable. *L. Törnqvist* (Helsinki).

Danø, Sven. Linear programming. Nordisk Mat. Tidskr. 4 (1956), 121–138, 175–176. (Danish. English summary)

A proof is given for the fact, that an optimal solution of a linear programming problem occurs at an extreme point of the set forming feasible solutions.

The article also describes the Simplex Method. The dual properties of programming problems are proven. *S. Malmquist* (Stockholm).

See also: Jessop, p. 675; Brown, p. 678; Nordbotten, p. 683.

Biology and Sociology

Krakau, C. E. T. Frequency analysis of neuronal time series. Kungl. Fysiogr. Sällsk. i Lund Forh. 26 (1956), no. 16, 20 pp.

This paper discusses the possibility of representing potential differences of the brain as stationary time series. The author uses as a model a linear process of the type encountered in shot noise theory. The form of the spectrum is discussed in some detail under various assumptions concerning the form and occurrence of the formations. *U. Grenander* (Stockholm).

Robson, D. S. Applications of the k_4 statistic to genetic variance component analyses. Biometrics 12 (1956), 433–444.

The distinguishing property of the k_4 statistic is that it estimates the fourth population cumulant. If $\{Y_i\}$ $(i=1, \cdots, n)$ is a sample of n independent observations on the chance variable Y then the statistic

$$k_4(Y_i)=\frac{n}{(n-1)(n-2)(n-3)} \times\left\{(n+1)\sum_1^n (Y_i-\bar{y})^4-\frac{3(n-1)}{n}\left[\sum_1^n (Y_i-\bar{y})^2\right]^2\right\}$$

has the expectation

$$Ek_4(Y_i)=E(Y-EY)^4-3[E(Y-EY)^2]^2=K_4(y).$$

This paper presents two essentially distinct applications of this property of the k_4 statistic to genetic variance component analysis, (i) in the unbiased estimation of the variance of estimates of variance components and (ii) in the estimation of the number of genes or factors controlling the inheritance of a quantitative character under the additive model with dominance. *W. T. Federer.*

Information and Communication Theory

Vallée, Robert. Un aspect du problème de l'observation. Methodos 7 (1955), 289–294.

The relation between a physical scalar field f and the observed field g may be represented by an operator A: $g=Af$. A class of operator A may form a commutative ring possessing divisors of zero. A number of physical examples are given to illustrate interpretations of sum and product. A sum is introduced to represent communication of information, one term of the sum representing "noise". *C. C. Torrance* (Monterey, Calif.).

★ **Herdan, G. Language as choice and chance.** P. Noordhoff N.V., Groningen, 1956. xiii+356 pp. $8.00.

The author's point of view is that the fact that an utterance consists of deliberately chosen words (the element of choice in the title) does not preclude its being a random sample (the element of chance in the title) with regard to some features. He is searching for such features which permit him to find regularities of statistical type; this does not mean, however, that we have just a conventional application of statistics — he tries to build up a statistical linguistics dealing with sui generis quantities. An important part of the discussion forms an attempt to "raise to quantitative level" the duality of "langue" and "parole" (introduced in the XIX century by de Saussure), the first corresponding to a statistical population and the second to a sample. This constitutes, however, according to the author, only one aspect of duality, a term to which he gives a very broad interpretation which permits him to cover by it dualities that occur in mathematics as well as many features of linguistics. In order not to encumber the text with technical details the author who is Lecturer in Statistics, University of Bristol, England, devotes the last 50 pages to a sketch of statistical theory. The book contains over a hundred numerical tables, several bibliographical lists totalling 200 titles, and very many misprints. *G. Y. Rainich.*

Control Systems

Bass, Robert W. A generalization of the functional relation $Y(t+s)=Y(t)\cdot Y(s)$ to piecewise-linear difference-differential equations. Quart. Appl. Math. **14** (1957), 415–417.

The author considers the difference-differential equation

$$y'=Ay+f[\Sigma\,(t-t_d)],\quad \Sigma\,(t)=b\cdot By,$$

where A and B are constant $n\times n$ matrices, y is a variable n-vector, b is a constant n-vector, and the dot indicates the scalar product. f is a vector step-function of the scalar Σ. Such equations arise in the theory of on-off control systems with time-delay. Assuming that the solutions of the given equation possess undesirable properties, whereas those of the undelayed equation

$$y'=Ay+f[\Sigma\,(t)]$$

do not, the author determines a modification of the given equation which is stated to have the effect of eliminating the undesirable properties. The discussion is quite brief, and not entirely clear. *L. A. MacColl.*

★Cypkin, Ya. Z. Teoriya releĭnyh sistem avtomatičeskogo regulirovaniya. [Theory of relay servomechanisms.] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955. 456 pp. 15.40 rubles.

This is a careful, detailed and exhaustive exposition of the theory of relay control systems. In the main, the discussion centers on the analysis of single loop systems comprising a relay and one more linear two-poles. Among the more important problems not treated in this text are those of the analysis of relay servomechanisms subjected to random inputs, the behavior of systems containing two or more relays, and the design of systems in which the relays control the parameters of linear components.

Chapter I surveys the various types of relay systems encountered in current practice and discusses their classification in terms of the characteristics of linear and nonlinear components. In Chapter II, the author shows how to set up the differential equations governing the behavior of a relay system. Chapter III is given over to a discussion of the response of a relay (with or without a dead zone) to a specified function of time. In Chapter IV, the notion of asymptotic stability is introduced and several stability criteria are formulated in terms of the transmission function and the impulsive response of the linear component. Chapters V, VI and VII deal exhaustively with the determination of the period, amplitude and other characteristics of both free and forced oscillations in relay systems. Here the author makes effective use of the hodograph function, $I(\omega)$, which is defined as

$$I(w)=-\frac{1}{w}z\left(\frac{\pi}{w}\right)-iz\left(\frac{\pi}{w}\right),$$

where $z(t)$ is the output of the linear component when the input to the relay is $x(t)=\sin\omega t$.

In Chapters VIII and IX, the author describes an interesting method for determining the stability of periodic regimes in relay systems. By expressing the derivative of the relay characteristic in terms of delta-functions, the equation satisfied by a perturbation, $\xi(t)$, is put into the form of a difference equation. In the case of a simple relay element, this amounts to identifying $\xi(t)$ with the output of a sampled-data system comprising an impulse modulator in tandem with the linear component of the relay system. Relay elements with dead zones and/or hysteresis give rise to more complicated types of sampled-data systems, but, in any case, the problem of determining the stability of a periodic regime is reduced to the much more tractable problem of analyzing a linear sampled-data transducer.

The last two chapters deal with the means of linearization of relay systems and with the optimization of relay controllers along the lines of the work of D. McDonald [Rev. Sci. Instrum. **23** (1952), 22–30], T. M. Stout [Proc. Nat. Electronics Conference **9** (1953), 188–198], and others.

Disregarding a few minor shortcomings, this book is undoubtedly an important contribution to the literature of automatic control systems. It contains a wealth of new material and is the most comprehensive of its kind to be published. *L. A. Zadeh* (Pelham, N.Y.).

HISTORY, BIOGRAPHY

★Eulerus, Leonhardus. Opera omnia. Series prima. Opera mathematica. Vol. XXIX. Commentationes geometricae. Vol. quartum. Edidit Andreas Speiser. Societas Scientiarum Naturalium Helveticae, Lausanne, 1956. xliv+448 pp.

This volume contains twenty papers and some fragments, all carefully edited by A. Speiser, with a full introduction, footnotes correcting errors, and an index. Some of the problems treated are: to find a plane curve which is similar to its first, second, or third evolute; to find a curve whose sectors (from a fixed origin and initial radius) are proportional to the squares of the corresponding arcs; to find the conditions for finite area between the curve

$$Ax^\alpha y^\beta+Bx^\gamma y^\delta=C$$

and its asymptotes; to find the ellipse of minimal area through three or four given points. The author's use of oblique axes suggests that he recognized the affine nature of these last problems. One of the most elegant passages is his proof of Lexell's theorem that, if A and B are two fixed points on a sphere, the locus of the third vertex of a spherical triangle ABC of constant area is a small circle.

H. S. M. Coxeter (Toronto, Ont.).

Halmos, P. R. "Nicolas Bourbaki." Scientific American, May 1957, 88–99.

Using this whimsical pseudonym, derived from the name of a general in the Franco-Prussian War, a group of young French mathematicians has written 20 volumes of an extraordinary treatise on mathematics.

Author's summary.

Aleksandrov, P.; Samarskiĭ, A.; and Svešnikov, A. Andreĭ Nikolaevič Tihonov (on the fiftieth anniversary of his birth). Uspehi Mat. Nauk (N.S.) 11 (1956), no. 6(72), 235–245 (1 plate). (Russian)

Kuroš, A. G. Obituary: Otto Yul'evič Šmidt. Uspehi Mat. Nauk (N.S.) 11 (1956), no. 6(72), 227–233 (1 plate). (Russian)

Natucci, Alpinolo. Che cosa contiene la "Nova Scientia" di Tartaglia? Giorn. Mat. Battaglini (5) 4(84) (1956), 261–271.

Statements, with short commentary, of the definitions and theorems of the three books of Tartaglia's Nova Scientia, first published in Venice in 1537.

Nevalinna, Rolf. Gauss and non-euclidean geometry. Nordisk Mat. Tidskr. 4 (1956), 195–209, 229. (Swedish. English summary)

Haimovici, A. Quelques remarques sur l'histoire des équations différentielles. Rev. Univ. "Al. I. Cuza" Inst. Politehn. Iaşi 1 (1954), 417–426. (Romanian)

The question which the author poses to himself in this note is: What can be said about the prehistory of the concept of differential equations? His answer to this question consists in references to some of the precursors of Leibniz and Newton. He asserts (it is true, without having convinced the referee of this) that one of the first differential equations was studied by Neper (of course, without the formal accomplishments of this concept and without consciousness of this). One can use two progressions, an arithmetic one and a geometric one, for the study of the logarithm. This resource was considered already by Chuquet (1484) and Stifel (1553), without proceeding to the calculus with logarithms. The essential achievement of Neper (1616) was, according to the author, that he studied the logarithm by replacing these two progressions by two movements .This was what makes the author assert that Neper was considering essentially the differential equation $dy/dx = -10^{-7}vy$ where v is a constant), whose solution is $x = 10^7 v^{-1} \log y^{-1}$. The author asserts besides this that also Galilei and Mariotte (1676) have considered "essentially" differential equations.

B. Germansky (Jerusalem).

Natucci, Alpinolo. Il calcolo dei radicali in Nicolò Tartaglia. Boll. Un. Mat. Ital. (3) 11 (1956), 594–598.

A short bibliographical note is followed by a detailed discussion of root extraction in the tenth book of Tartaglia's General Trattato.

In memoria di Bertrand Gambier. Boll. Un. Mat. Ital. (3) 11 (1956), 599–607.

A list of published works.

Villa, M. Ricerche di geometria differenziale in Romania. Boll. Un. Mat. Ital. (3) 11 (1956), 591–593.

A short history of differential geometry in Romania.

Singh, A. N. Hindu superiority in mathematics. Bull. Allahabad Univ. Math. Assoc. 16 (1951–56), 11–30.

Historical remarks under the headings: decimal place-value notation, arithmetic, algebra, geometry, trigonometry, calculus, finite and infinite series.

Shankar Shukla, Kripa. On three stanzas from the Pañca-siddhāntikā. Ganita 5 (1954), 129–136 (1955).

Three stanzas from the Pañca-siddhāntikā of Varāhamihira (c. 550 A.D.), edited by G. Thibaut and S. Dvivedī (1889 A.D.) are in disagreement with other ancient Hindu authorities concerning the distances from the Sun at which the planets become visible and also concerning the distances and diameters of the Sun and the Moon. It was discovered by the present author that this disagreement arises solely from emendations made by Thibaut and Dvivedī in the traditional text of the Pañca-siddhāntikā.

Shankar Shukla, Kripa. A note on the Rāja-mṛgāṅka of Bhoja published by the Adyar Library. Ganita 5 (1954), 149–151 (1955).

It is made clear that this edition of the Rāja-mṛgāṅka is not the original text of Bhoja but an abridged edition by some later writer.

Popa, Ilie. Alexander and Vera Myller. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I. (N.S.) 1 (1955), xi–xxiv. (Romanian)

Biographical sketches with portraits of a well-known couple of Romanian mathematicians who were both students of Hilbert in the beginning of the century.

Sarton, George. Raymond Clare Archibald. Osiris 12 (1956), 5–34 (1 plate).

A general biography with a list of published works. The volume is dedicated to the memory of Archibald.

Ślebodziński, W. Scientific work of K. Żorawski. Prace Mat. 2 (1956), 79–93. (Polish)

Hodge, W. V. D. Obituary: Henry Frederick Baker. J. London Math. Soc. 32 (1957), 112–128.

A general and scientific biography, with a list of articles and books.

Edge, W. L. Obituary: Miss C. M. Hamill. Edinburgh Math. Notes. no. 40 (1956), 22–25.

A general and scientific biography.

Houstoun, R. A. Obituary: Horatio S. Carslaw. Edinburgh Math. Notes. no. 40 (1956), 26.

Biographical details of a general nature.

Sinvhal, S. D. Dr. Avadhesh Narain Singh (a life sketch). Ganita 5 (1954), i–vii (1 plate) (1955).

A general biography with a list of 44 published works.

★ **Pantazi, A. Opera matematică. [Mathematical works.]** Editura Academiei Republicii Populare Romîne, Bucureşti, 1956. 497 pp. 24,20 lei.